A solution to the Maxwell equations

Griffiths, ***Electrodynamics*****, fourth edition, problem 7.37**

Recall that $\theta(vt - r) = 0$ when $r \geq vt$ and that

$$\frac{d\theta(x)}{dx} = \delta(x).$$

The situation described is: A charge of $+q$ sits directly on top of a charge of $-q$. At $t = 0$, the charge $+q$ explodes into a spherical shell expanding at speed v. This shell, of course, has radius $R = vt$, surface charge density $\frac{q}{4\pi R^2}$, volume charge density $\frac{q}{4\pi R^2}\delta(R-r)$, and current density $v\frac{q}{4\pi R^2}\delta(R-r)\hat{r}$.

From $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ we conclude (spherical coordinates):

$$\begin{aligned}
\frac{\rho}{\epsilon_0} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(-\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\theta(vt-r)\right)\right] \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(-\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)\right]\theta(vt-r) + \frac{1}{r^2}\left[r^2\left(-\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)\right]\frac{\partial}{\partial r}\theta(vt-r) \\
&= -\frac{q}{\epsilon_0}\delta^{(3)}(\vec{r})\theta(vt-r) - \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\delta(vt-r)(-1) \\
\rho &= -q\delta^{(3)}(\vec{r}) + \frac{q}{4\pi(vt)^2}\delta(vt-r)
\end{aligned}$$

From $\nabla \times \vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$ we conclude:

$$\begin{aligned}
\vec{J} &= -\epsilon_0\frac{\partial\vec{E}}{\partial t} \\
&= -\epsilon_0\left(-\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)\hat{r}\frac{\partial\theta(vt-r)}{\partial t} \\
&= \frac{1}{4\pi}\frac{q}{r^2}\hat{r}\delta(vt-r)v \\
&= v\frac{q}{4\pi(vt)^2}\delta(vt-r)\hat{r}
\end{aligned}$$

To assure that this is a solution, we need only check that

$$\nabla \cdot \vec{B} = 0,$$

which is obviously true, and that

$$\nabla \times \vec{E} = -\frac{\partial\vec{B}}{\partial t}.$$

In spherical coordinates, for a radially symmetric vector $\vec{E} = E(r)\hat{r}$, the curl is

$$\nabla \times \vec{E} = 0\,\hat{r} + \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial E(r)}{\partial\phi}\hat{\theta} + \frac{1}{r}\left(-\frac{\partial E(r)}{\partial\theta}\right)\hat{\phi} = 0,$$

so this is true too.