Electrostatics of a circuit

Griffiths, *Electrodynamics*, fourth edition, problem 7.42

(a) Find the potential.

The potential everywhere is the solution to

$$\nabla^2 V(s,\phi) = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

subject to boundary condition

$$V(a,\phi) = \frac{V_0 \phi}{2\pi}.$$

Use separation of variables:

$$V(s,\phi) = S(s)\Phi(\phi)$$

so that

$$\begin{aligned}
\frac{1}{s}\frac{\partial}{\partial s}\left(sS'(s)\Phi(\phi)\right) + \frac{1}{s^2}S(s)\Phi''(\phi) &= 0 \\
s(S'(s) + sS''(s))\Phi(\phi) + S(s)\Phi''(\phi) &= 0 \\
\frac{s(S'(s) + sS''(s))}{S(s)} &= -\frac{\Phi''(\phi)}{\Phi(\phi)}.
\end{aligned}$$

Through the standard separation-of-variables argument, the left-hand side depends only on the variable s, while the right-hand side depends only on the variable ϕ, and s is independent of ϕ, so both sides must be equal to the constant C.

The angular equation is

$$\Phi''(\phi) = -C\Phi(\phi)$$

with solution

$$\Phi(\phi) = A\sin(\omega\phi) + B\cos(\omega\phi) \qquad \text{where} \qquad C = \omega^2.$$

In order for this solution to obey the necessary condition

$$\Phi(\phi) = \Phi(\phi + 2\pi)$$

for all values of ϕ, we must have

$$\omega = 0, 1, 2, 3, \ldots.$$

The radial equation is then

$$\begin{aligned}
\frac{s(S'(s) + sS''(s))}{S(s)} &= C = \omega^2 \\
s^2 S''(s) + sS'(s) &= \omega^2 S(s)
\end{aligned}$$

A little playing around with trial solution s^α shows that the solution is

$$S(s) = As^\omega + B\frac{1}{s^\omega}.$$

Clearly, inside the cylinder we need to use s^{ω} while outside we need to use $s^{-\omega}$.

To summarize, the general solution inside the cylinder is

$$V(s,\phi) = \sum_{n=0}^{\infty} s^n [A_n \sin(n\phi) + B_n \cos(n\phi)],$$

while the general solution outside the cylinder is

$$V(s,\phi) = \sum_{n=0}^{\infty} \frac{1}{s^n} [A'_n \sin(n\phi) + B'_n \cos(n\phi)].$$

Our task now is to find the coefficients A_n and B_n, as well as A'_n and B'_n, by fitting to the boundary condition at $s = a$:

$$V(a,\phi) = \frac{V_0 \phi}{2\pi} \quad \text{for} \quad -\pi < \phi < \pi.$$

We could do this in the usual way, using the orthogonality of the sines and cosines, in the same way that one finds Fourier series. Alternatively you might just happen to know this result (which is the Fourier series representation of a sawtooth signal), published in H.D. Dwight, *Tables of Integrals* (1961), equation 416.07:

$$\phi = -2 \sum_{m=1}^{\infty} \frac{(-1)^m \sin(m\phi)}{m} \quad \text{for} \quad -\pi < \phi < \pi.$$

This means that the boundary condition is

$$V(a,\phi) = -\frac{V_0}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m \sin(m\phi)}{m},$$

so the solution inside the cylinder is

$$V(s,\phi) = -\frac{V_0}{\pi} \sum_{n=1}^{\infty} \left(-\frac{s}{a}\right)^n \frac{\sin(n\phi)}{n},$$

while the solution outside the cylinder is

$$V(s,\phi) = -\frac{V_0}{\pi} \sum_{n=1}^{\infty} \left(-\frac{a}{s}\right)^n \frac{\sin(n\phi)}{n}.$$

These last two expressions are perfectly correct solutions, but they are rendered a little more convenient by the result published in L.B.W. Jolley, *Summation of Series* (1961), equation 540:

$$\sum_{n=1}^{\infty} \frac{A^n \sin(n\theta)}{n} = \operatorname{atan}\left(\frac{A\sin\theta}{1 - A\cos\theta}\right) \quad \text{where} \quad A^2 < 1.$$

Using this result, the solution inside the cylinder is

$$V(s,\phi) = \frac{V_0}{\pi} \operatorname{atan}\left(\frac{s\sin\phi}{a + s\cos\phi}\right),$$

while the solution outside the cylinder is

$$V(s,\phi) = \frac{V_0}{\pi}\,\text{atan}\left(\frac{a\sin\phi}{s + a\cos\phi}\right).$$

⟦It used to be difficult to sum infinite series, so it was an important physics skill to be able to get your own infinite series and express it in terms of already-tabulated functions like $\sin(\theta)$ and $\text{atan}(x)$. This skill is less important today, but it still leads to a sort of elegance.⟧

(b) Find the surface charge density.

The surface charge density is

$$\sigma(\vec{r}) = \epsilon_0 E_\perp(\vec{r})$$

evaluated at a point $\vec{r}$ on the surface, where $E_\perp(\vec{r})$ is the component of $\vec{E}(\vec{r})$ perpendicular to the surface at that point. Meanwhile

$$\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r}) = -\frac{\partial V}{\partial s}\vec{s} - \frac{1}{s}\frac{\partial V}{\partial \phi}\vec{\phi},$$

so

$$E_\perp(\vec{r}) = -\frac{\partial V}{\partial s}.$$

We can take this derivative either inside or outside... we'll get the same answer at the surface. I chose to take it for the outside because the variable s occurs fewer times in that expression.

$$\begin{aligned}
V(s,\phi) &= \frac{V_0}{\pi}\,\text{atan}\left(\frac{a\sin\phi}{s + a\cos\phi}\right) \\
\frac{\partial V}{\partial s} &= \frac{V_0}{\pi}\left[\frac{1}{1 + \left(\dfrac{a\sin\phi}{s + a\cos\phi}\right)^2}\right]\left[-\frac{a\sin\phi}{(s + a\cos\phi)^2}\right] \\
&= \frac{V_0}{\pi}\left[\frac{(s + a\cos\phi)^2}{(s + a\cos\phi)^2 + (a\sin\phi)^2}\right]\left[-\frac{a\sin\phi}{(s + a\cos\phi)^2}\right] \\
&= -\frac{V_0}{\pi}\frac{a\sin\phi}{s^2 + 2sa\cos\phi + a^2}.
\end{aligned}$$

So at the surface, $s = a$,

$$\frac{\partial V}{\partial s} = -\frac{V_0}{\pi}\frac{a\sin\phi}{a^2 + 2a^2\cos\phi + a^2} = -\frac{V_0}{\pi}\frac{1}{2a}\frac{\sin\phi}{1 + \cos\phi} = -\frac{V_0}{2\pi a}\tan(\phi/2),$$

where the last step uses the tangent half-angle formula Dwight equation 406.2.

Wrapping up,

$$\sigma = \frac{\epsilon_0 V_0}{2\pi a}\tan(\phi/2).$$