Time averages

Griffiths, *Electrodynamics*, fourth edition, problem 9.12

We have

$$f(\vec{r},t) = A\cos(\vec{k}\cdot\vec{r} - \omega t + \delta_a) = \Re e\{Ae^{i(\vec{k}\cdot\vec{r}-\omega t+\delta_a)}\} = \Re e\{\tilde{f}e^{i(\vec{k}\cdot\vec{r}-\omega t)}\}$$

where $\tilde{f} = Ae^{i\delta_a}$. And we have

$$g(\vec{r},t) = B\cos(\vec{k}\cdot\vec{r} - \omega t + \delta_b) = \Re e\{Be^{i(\vec{k}\cdot\vec{r}-\omega t+\delta_b)}\} = \Re e\{\tilde{g}e^{i(\vec{k}\cdot\vec{r}-\omega t)}\}$$

where $\tilde{g} = Be^{i\delta_b}$.

But $\Re e\{z\} = \frac{1}{2}[z + z^*]$, so

$$\begin{aligned} f(\vec{r},t) &= \tfrac{1}{2}[\tilde{f}e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \tilde{f}^* e^{-i(\vec{k}\cdot\vec{r}-\omega t)}] \\ g(\vec{r},t) &= \tfrac{1}{2}[\tilde{g}e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \tilde{g}^* e^{-i(\vec{k}\cdot\vec{r}-\omega t)}] \end{aligned}$$

and

$$fg = \tfrac{1}{4}[\tilde{f}\tilde{g}e^{2i(\vec{k}\cdot\vec{r}-\omega t)} + \tilde{f}\tilde{g}^* + \tilde{f}^*\tilde{g} + \tilde{f}^*\tilde{g}^* e^{-2i(\vec{k}\cdot\vec{r}-\omega t)}].$$

Now for time averages:

$$\begin{aligned} \langle e^{2i(\vec{k}\cdot\vec{r}-\omega t)}\rangle &= \frac{1}{T}\int_0^T e^{2i(\vec{k}\cdot\vec{r}-\omega t)}dt = \frac{1}{T}e^{2i\vec{k}\cdot\vec{r}}\int_0^T e^{-2i\omega t}dt = \frac{1}{T}e^{2i\vec{k}\cdot\vec{r}}\left[\frac{1}{-2i\omega}e^{-2i\omega t}\right]_0^T \\ &= \frac{1}{-2i\omega T}e^{2i\vec{k}\cdot\vec{r}}\left[1 - e^{-2i\omega T}\right] = \frac{1}{-2i(2\pi)}e^{2i\vec{k}\cdot\vec{r}}\left[1 - e^{-2i(2\pi)}\right] = 0 \end{aligned}$$

and, clearly

$$\langle e^{-2i(\vec{k}\cdot\vec{r}-\omega t)}\rangle = 0$$

also. Thus

$$\langle fg\rangle = \tfrac{1}{4}[\tilde{f}\tilde{g}^* + \tilde{f}^*\tilde{g}] = \tfrac{1}{2}\Re e\{\tilde{f}\tilde{g}^*\} = \tfrac{1}{2}AB\cos(\delta_a - \delta_b).$$