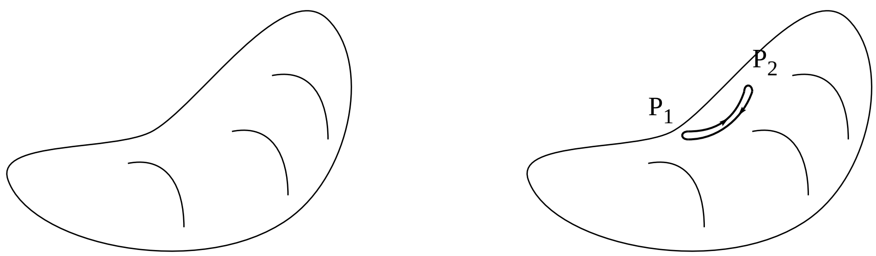
The divergence of the curl is zero

(Approach from Purcell, *Electricity and Magnetism*, problem 2.15.)

If $\vec{A}(\vec{r})$ is a vector field with continuous derivatives, then

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}(\vec{r})) = 0.$$

How to prove? You could plug-and-chug in Cartesian coordinates. But it's easier and more insightful to do it this way.



Consider (figure on the left) the volume $\mathcal{V}$ enclosed by surface $\mathcal{S}$. Apply the divergence theorem to function $\vec{F}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$, giving

$$\int_{\mathcal{V}} \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}(\vec{r}))\, d^3r = \int_{\mathcal{S} \text{ of } \mathcal{V}} (\vec{\nabla} \times \vec{A}(\vec{r})) \cdot \hat{n}\, dA.$$

Now slice out and remove from the surface a tiny sliver (figure on the right). Technically we've altered $\mathcal{S}$, but this tiny alteration will not affect the value of the surface integral. The edge $\mathcal{E}$ of the altered $\mathcal{S}$ is the edge of the sliver. Apply the circulation theorem to $\vec{F}(\vec{r}) = \vec{A}(\vec{r})$, giving

$$\int_{\mathcal{S}} (\vec{\nabla} \times \vec{A}(\vec{r})) \cdot \hat{n}\, dA = \int_{\mathcal{E} \text{ of } \mathcal{S}} \vec{A}(\vec{r}) \cdot d\vec{\ell}.$$

But

$$\int_{\mathcal{E} \text{ of } \mathcal{S}} \vec{A} \cdot d\vec{\ell} = \int_{P_1 \text{ to } P_2} \vec{A} \cdot d\vec{\ell} + \int_{P_2 \text{ to } P_1} \vec{A} \cdot d\vec{\ell} = \int_{P_1 \text{ to } P_2} \vec{A} \cdot d\vec{\ell} - \int_{P_1 \text{ to } P_2} \vec{A} \cdot d\vec{\ell} = 0.$$

So for any volume $\mathcal{V}$,

$$\int_{\mathcal{V}} \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}(\vec{r}))\, d^3r = 0.$$

Because this holds for *any* volume,

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}(\vec{r})) = 0.$$

(This is an example of the theorem that "the boundary of a boundary is zero," as emphasized by Misner, Thorne, and Wheeler, *Gravitation*, box 15.1, pages 365–371.)