The Maxwell stress tensor

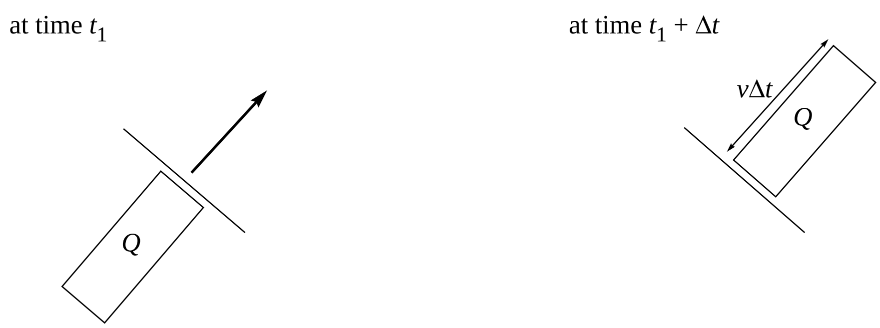
Dan Styer, 18 April 2012; revised 4 November 2016, 19 January 2017

⟦To accompany David J. Griffiths, *Introduction to Electrodynamics*, fourth edition (2013).⟧

We begin with the familiar: the flow of charge density, which is described through current density — a vector. After this review, we will generalize to the flow of momentum density, which (we will find) is described through the Maxwell stress tensor.

Charge transport

Suppose charge flows uniformly in space and time, like a broad steady wind.



A plug of length $v\Delta t$, cross-sectional area $A_\perp$, passes through the imaginary plane perpendicular to the "wind". This plug contains charge

$$Q = \rho|\vec{v}|\Delta t A_\perp,$$

so the current density, namely

$$\frac{\text{charge/time}}{\text{cross-sectional area}}$$

has direction parallel to $\vec{v}$ and magnitude

$$|\vec{J}| = \frac{Q/\Delta t}{A_\perp} = \frac{\rho|\vec{v}|\Delta t A_\perp/\Delta t}{A_\perp} = \rho|\vec{v}|,$$

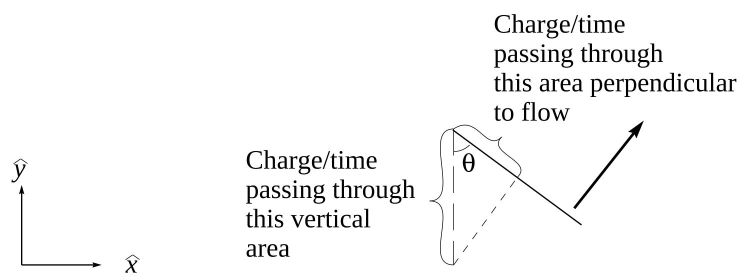
whence

$$\vec{J} = \rho\vec{v}.$$

Suppose you had a current detector shaped like a tennis racquet, which measured the current (charge per time) passing through the webbing of the racquet. If the webbing were oriented parallel to the charge flow, then the detector would read zero current. You you changed its orientation to the plane facing the flow, then the detector reading would be a maximum. This is how you could measure the current density: twist your racquet detector this way and that until its reading is a maximum, then the current density vector is oriented perpendicular to the webbing, and has magnitude of the current reading divided by the area of the webbing.

While this technique works, it's inefficient. You have to take many readings, homing in on a maximum. You have to twist your body like a contortionist in order to check out various orientations. Isn't there an easier way?

There is. In fact, you can find the current density vector by taking just three readings, one with the racquet webbing held perpendicular to $\hat{x}$, one with the racquet webbing held perpendicular to $\hat{y}$, and one with the racquet webbing held perpendicular to $\hat{z}$. The first such experiment is illustrated below:



Suppose the marked-off part of the vertical plane has area A_x. Then the marked-off part of the perpendicular-to-flow plane has area $A_x \cos\theta$. The two areas are equal in charge/time passing through, namely

$$|\vec{J}|A_x \cos\theta = J_x A_x = \vec{J}\cdot\hat{x}\, A_x.$$

(In general, the current flowing through area A perpendicular to $\hat{n}$ is $\vec{J}\cdot\hat{n}\, A$.) Thus, if you hold the racquet detector with webbing perpendicular to $\hat{x}$, find the current passing through, and divide that current by the area, you will find the x-component of current density, which we call J_x. Similarly you can measure the y and z components, and with just these three measurements you will find the current density vector

$$\vec{J} \doteq (J_x, J_y, J_z).$$

Notice that we never experimentally measure a vector: all our measurements are of components of a vector projected on various unit vectors. By putting together a series of such measurements (either through three components projected onto the basis vectors, or through homing in on the maximum), we can uncover the vector.

I hope that nothing I've said in this section is new to you. I just wanted to remind you of how current flow is defined and what it means to be a vector.

Momentum transport

Suppose momentum flows uniformly in space and time, like a broad steady wind. If this momentum were carried by matter, like a literal wind or rain, then the motion of the air or raindrops ($\vec{v}$) would be parallel to the momentum being transported ($\vec{p}$). But in the case of electromagnetic fields the momentum might or might not be parallel to the direction the field is moving, and the diagram below admits that possibility.

The reasoning proceeds exactly as it did for charge transport, except that charge density ρ is replaced by momentum density $\vec{g}$. A plug of length $v\Delta t$, cross-sectional area $A_\perp$, passes through the imaginary plane perpendicular to the "wind". This plug contains momentum

$$\vec{p} = \vec{g}|\vec{v}|\Delta t A_\perp,$$

so the current density of x-momentum, namely

$$\frac{x\text{-momentum/time}}{\text{cross-sectional area}}$$

has direction parallel to $\vec{v}$ and magnitude

$$\frac{p_x/\Delta t}{A_\perp} = \frac{g_x|\vec{v}|\Delta t A_\perp/\Delta t}{A_\perp} = g_x|\vec{v}|,$$

whence the current density of x-momentum is

$$g_x\vec{v}.$$

Of course, you can do the same calculation for y-momentum density and z-momentum density. In total, the momentum current density is

$$\vec{g}\vec{v}.$$

What is this thing we get by multiplying two vectors in this way? It's not a scalar, which would result from a dot product, and it's not a vector, which would result from a cross product. This is called an "outer product" or a "tensor product" and the result is called a tensor. James Clerk Maxwell recognized the importance of this particular combination, except that he defined it with a negative sign. The "Maxwell stress tensor" is defined through

$$\overleftrightarrow{T} = -\vec{g}\vec{v}.$$

The physical meaning of the Maxwell stress tensor is exactly as described above. You could image a momentum detector shaped like a tennis racquet, and everything we said above about charge current density measurements (charge current density is a vector) would apply to momentum current density measurements (momentum current density is a tensor). In particular, if you look at components in some given basis you'll find

$$\overset{\leftrightarrow}{T} \doteq \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix} = -\begin{pmatrix} g_x \\ g_y \\ g_z \end{pmatrix} \begin{pmatrix} v_x & v_y & v_z \end{pmatrix} = -\begin{pmatrix} g_x v_x & g_x v_y & g_x v_z \\ g_y v_x & g_y v_y & g_y v_z \\ g_z v_x & g_z v_y & g_z v_z \end{pmatrix}.$$

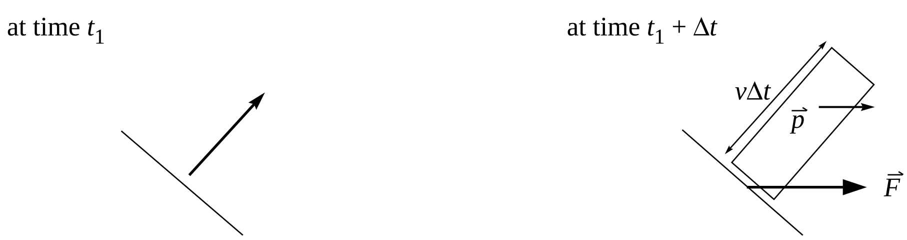
Thus, for example, $-T_{xz}$ represents the current of x-momentum passing through a plane perpendicular to z.

As with vectors, we never experimentally measure tensors: all our measurements are of components of tensors. And as with vectors, this doesn't mean that tensors are useless or unnatural. It just means that they're less familiar.

Exercise: Suppose the material is indeed, say, wind, so that the momentum transported is always in the direction of the motion of the material flow. Is the tensor then represented by a diagonal matrix? Symmetric? Antisymmetric? What would the tensor components look like if the basis were oriented so that the x direction points along the direction of the wind?

Force

We've considered the flow of momentum, carried in a field, across a plane. But there's another possibility. Remember that momentum is conserved only in the absence of external force. We could start with nothing, and end up with a plug of momentum, by exerting a force.



In this figure, think of the lower left portion as occupied by some charges and currents. After some time has passed, there is a plug of fields on the upper right. The force exerted by these charges and currents on the field, in order to produce this momentum, is

$$\vec{F} = \frac{\vec{p}}{\Delta t} = \frac{\vec{g}|\vec{v}|\Delta t A_\perp}{\Delta t} = \vec{g}\vec{v}\cdot\hat{n}\, A_\perp = -\overset{\leftrightarrow}{T}\cdot\hat{n}\, A_\perp.$$

By Newton's third law, the force on the charges and currents by the field is thus

$$\vec{F} = \overset{\leftrightarrow}{T}\cdot\hat{n}\, A_\perp.$$

If the surface is not uniform, then the total force on the charges and currents is found by integrating over the surface

$$\vec{F}_{\text{total}} = \int_{\mathcal{S}} \overleftrightarrow{T} \cdot \hat{n}\, dA$$

Cute trick! Instead of integrating the force density over the volume of the charges and currents, you only need to surface integrate the Maxwell stress tensor over the surface.

Connection to Griffiths

The motivation of the Maxwell stress tensor given here differs dramatically from the one given by Griffiths. I think that this way carries a lot more insight, which is why I use it. But Griffiths's approach has two advantages. First, my way holds only for static situations. (I said "steady wind" in the first sentence.) For a non-static situation, the total force on the charges and currents is (Griffiths equation 8.28)

$$\vec{F}_{\text{total}} = \int_{\mathcal{S}} \overleftrightarrow{T} \cdot \hat{n}\, dA - \epsilon_0 \mu_0 \frac{d}{dt} \int_{\mathcal{V}} \vec{S}\, d^3r,$$

where $\vec{S}$ is the Poynting vector.

Griffiths's second advantage is that in my approach, everything depends on "the velocity of the fields". A nice, compact concept, but how are you supposed to calculate it?! Griffiths gives a formula (equation 8.17) for the stress tensor in terms of fields alone:

$$\overleftrightarrow{T} = \epsilon_0 \left(\vec{E}\vec{E} - \tfrac{1}{2} E^2 \overleftrightarrow{I} \right) + \frac{1}{\mu_0} \left(\vec{B}\vec{B} - \tfrac{1}{2} B^2 \overleftrightarrow{I} \right),$$

where $\overleftrightarrow{I}$ represents the identity tensor, with components δ_{ij}.

Generalizations

The flow of a scalar (like charge density ρ) is described through a vector (like current density $\vec{J}$).

The flow of a vector (like momentum density $\vec{g}$) is described through a tensor (like the negative of the Maxwell stress tensor $-\overleftrightarrow{T}$).

What mathematical tool would one use to describe the flow of a tensor?

I ask this question not to make your brain hurt, but to open your mind to more and richer possibilities. The tensor that we've discussed, namely the Maxwell stress tensor, is an example of a "rank-2 tensor". In three dimensions, a rank-2 tensor can be described using 9 projections, called components, which are conveniently presented in a 3×3 matrix. The flow of a rank-2 tensor is described through a "rank-3 tensor". In three dimensions, a rank-3 tensor can be described using 27 components, and there's no real convenient way to present one on flat paper. (Sometimes I use a stack of three index cards, on each of which I write a 3×3 matrix. But even this is not really effective.) From this point of view, a vector is a rank-1 tensor and a scalar is a rank-0 tensor. In general, a rank-r tensor in d dimensions is specified through d^r components.

People get upset by the word tensor, but they shouldn't:

The motion of a point is described by a vector
The motion of a vector is described by a (rank-2) tensor
The motion of a (rank-2) tensor is described by a rank-3 tensor

and so forth. It's nothing more than that.