Classical action for the harmonic oscillator: Feynman-Hibbs problem 2-2

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Solution to problem 2-2 in *Quantum Mechanics and Path Integrals* by Richard P. Feynman and Albert R. Hibbs (emended edition, 2005).

The motion for the harmonic oscillator is of course known to be

$$x(t) = A\sin(\omega t) + B\cos(\omega t). \tag{1}$$

Fitting the boundary conditions $x(0) = x_a$ and $x(T) = x_b$ gives

$$A = \frac{x_b - x_a\cos(\omega T)}{\sin(\omega T)}, \qquad B = x_a. \tag{2}$$

From this, one could simply plug into the definition of action, work the time integrals, and get the result. I've done this, and it's not impossible, but it's far easier to follow the hint on page 363.

For the harmonic oscillator,

$$S_{cl} = \frac{m}{2}\int_{t_a}^{t_b} (\dot{x}^2 - \omega^2 x^2)\, dt. \tag{3}$$

We execute the first part of the integral using parts:

$$\begin{aligned}\int_{t_a}^{t_b} \dot{x}^2\, dt &= \int_{t_a}^{t_b} \dot{x}\dot{x}\, dt \\ &= \Big[x\dot{x}\Big]_{t_a}^{t_b} - \int_{t_a}^{t_b} x\ddot{x}\, dt.\end{aligned}$$

In most cases, this would be no help at all. But for the harmonic oscillator, $ma = -kx$, or $\ddot{x} = -\omega^2 x$, whence

$$\int_{t_a}^{t_b} \dot{x}^2\, dt = \Big[x\dot{x}\Big]_{t_a}^{t_b} + \omega^2\int_{t_a}^{t_b} xx\, dt,$$

and the total action is

$$S_{cl} = \frac{m}{2}\left[[x\dot{x}]_{t_a}^{t_b} + \omega^2\int_{t_a}^{t_b} xx\, dt - \omega^2\int_{t_a}^{t_b} x^2\, dt\right] = \frac{m}{2}\,[x\dot{x}]_{t_a}^{t_b}\,.$$

Writing the result more formally, we have the useful theorem that, for a harmonic oscillator,

$$S_{cl} = \frac{m}{2}\,[x(t)\dot{x}(t)]_{t_a}^{t_b}\,. \tag{4}$$

Now we return to the problem at hand.

$$S_{cl} = \frac{m}{2}\,[x(t)\dot{x}(t)]_0^T = \frac{m}{2}\,[x(T)\dot{x}(T) - x(0)\dot{x}(0)] = \frac{m}{2}\,[x_b\dot{x}(T) - x_a\dot{x}(0)]\,. \tag{5}$$

Given the explicit forms in equations (1) and (2), it's clear that

$$\dot{x}(t) = \omega\frac{x_b - x_a\cos(\omega T)}{\sin(\omega T)}\cos(\omega t) - \omega x_a \sin(\omega t), \tag{6}$$

whence

$$\begin{aligned}
\dot{x}(T) &= \omega\frac{x_b - x_a\cos(\omega T)}{\sin(\omega T)}\cos(\omega T) - \omega x_a \sin(\omega T), &(7)\\
\dot{x}(0) &= \omega\frac{x_b - x_a\cos(\omega T)}{\sin(\omega T)}. &(8)
\end{aligned}$$

At this point, it's all over but the substituting:

$$\begin{aligned}
S_{cl} &= \frac{m}{2}\left[x_b\dot{x}(T) - x_a\dot{x}(0)\right]\\
&= \frac{m}{2}\omega\left[x_b\frac{x_b - x_a\cos(\omega T)}{\sin(\omega T)}\cos(\omega T) - x_b x_a\sin(\omega T) - x_a\frac{x_b - x_a\cos(\omega T)}{\sin(\omega T)}\right]\\
&= \frac{m\omega}{2\sin(\omega T)}\left[x_b[x_b - x_a\cos(\omega T)]\cos(\omega T) - x_b x_a\sin^2(\omega T) - x_a[x_b - x_a\cos(\omega T)]\right]\\
&= \frac{m\omega}{2\sin(\omega T)}\left[x_b^2\cos(\omega T) - x_b x_a\cos^2(\omega T) - x_b x_a\sin^2(\omega T) - x_a x_b + x_a^2\cos(\omega T)\right]\\
&= \frac{m\omega}{2\sin(\omega T)}\left[(x_b^2 + x_a^2)\cos(\omega T) - 2x_b x_a\right]. &(9)
\end{aligned}$$