Notes on

Relativistic Dynamics

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Preface

These notes assume a knowledge of space and time in special relativity, and of force, energy, and momentum in classical mechanics (both at the college freshman level). They build on that knowledge to describe force, energy, and momentum in special relativity. These notes also use a few ideas from freshman-level electricity and magnetism, but not in an essential way. The intent is to present physical questions and their direct and straightforward (if laborious) solutions, rather than to show off how mathematically clever the author is.

Teaching notes: I use these notes over five or six lectures to college sophomores. On the first day I ask students what they remember about space and time in special relativity. Students are often surprised and gratified that they remember anything at all about such a counterintuitive subject.

Then I present "why we need relativistic dynamics" (section 2.1): if classical mechanics were correct, then a particle dropping from rest would achieve speeds greater than that of light. This is followed by one of the two "momentum motivations", either the collision motivation (sections 2.3, 2.4, and 2.5) or the four-vector motivation (sections 3.1, 3.2 and 3.3). I leave the other motivation for reading. I've tried it both ways and it doesn't seem to make any difference in how well the students learn. In either case I end up interpreting the "new quantity" $mc^2/\sqrt{1-(v/c)^2}$ (section 2.6) in class.

In class I present chapters 4 and 6, leaving chapter 5 for reading. It is impossible to overemphasize that the mass of a composite might not equal the sum of the masses of its constituents, which is why I make that point twice (once in section 4.1, again in section 4.2). I end with sections 7.1 and 7.2, "the effect of a force" and "starting from rest with a single constant force". This way we end by answering the question we started with, which ties the whole subject together.

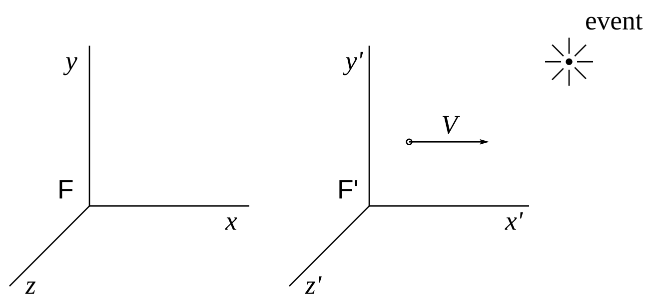
At this point, students are mentally exhausted: so many new and counterintuitive ideas, so close together. So I have never covered chapters 8 or 9. I just hope that when the students regain their footing they will look back at those two final chapters to learn things both wonderful and profound.

Acknowledgment: The discussion of hard-sphere forces in section 9.2 arose from a question by David Carr, a Ph.D. student in computer science at Charles Sturt University in Australia, who was designing a game to teach relativity.

Chapter 1

Space and Time

What do we know about space and time in special relativity?



Suppose an event happens. That event has space-time coordinates (x, y, z, t) in inertial frame F and space-time coordinates (x', y', z', t') in inertial frame F'. If frame F' moves at constant speed V relative to frame F, and the two frames coincide at time $t = t' = 0$, then we know that the two sets of coordinates are related through the Lorentz transformation:

$$\begin{aligned} x' &= \frac{x - Vt}{\sqrt{1-(V/c)^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - Vx/c^2}{\sqrt{1-(V/c)^2}} \end{aligned} \tag{1.1}$$

Some consequences of the Lorentz transformation are:

- *Classical limit.* If $V \ll c$, the Lorentz transformation is approximated by the common-sense Galilean transformation:

$$\begin{aligned} x' &= x - Vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \tag{1.2}$$

- *The invariant interval.* Although the coordinates of an event are different in the two frames, the combination

$$(ct)^2 - (x^2 + y^2 + z^2) \tag{1.3}$$

  is the same in all frames. This combination is called the "invariant interval" (or, rarely, the "Lorentz interval").

- *Lorentz transformation for differences.* If we consider the difference between two events, the coordinates are related through

$$\begin{aligned} \Delta x' &= \frac{\Delta x - V\Delta t}{\sqrt{1-(V/c)^2}} \\ \Delta y' &= \Delta y \\ \Delta z' &= \Delta z \\ \Delta t' &= \frac{\Delta t - V\Delta x/c^2}{\sqrt{1-(V/c)^2}} \end{aligned} \tag{1.4}$$

- *Relativity of simultaneity.* Two events simultaneous in one reference frame ($\Delta t = 0$) are not simultaneous in another ($\Delta t' = -(V\Delta x/c^2)/\sqrt{1-(V/c)^2}$ ... the rear event happens first).

- *Time dilation.* A moving clock ticks slowly.

- *Length contraction.* A moving rod is short.

- *Speed addition.* If a bird travels in the x-direction with speed v_b in frame F, then its speed in frame F' is

$$v_b' = \frac{v_b - V}{1 - v_b V/c^2}. \tag{1.5}$$

- *The speed of light* is the same in all inertial frames.

- *Speed limit.* No message can travel faster than light (in any inertial frame).

- *No material is completely rigid.*

Some people look at these consequences and make an additional conclusion: "Space and time are all fucked up." That's wrong. The proper conclusions are that "Space and time don't adhere to common sense" and that "Common sense is all fucked up." It is our duty as scientists to change our minds to fit nature, not to change nature to fit the preconceptions in our minds.

Problems

1.1. *He says, she says.* Veronica speeds past Ivan. He says her clocks tick slowly, she says his clocks tick slowly. This is not a logical contradiction because…

a. Ivan sees the hands of Veronica's clocks as length contracted.

b. Veronica compares her clock to two of Ivan's clocks, and those two clocks aren't synchronized.

c. two events simultaneous in Ivan's frame are always simultaneous in Veronica's frame as well.

d. a moving rod is short.

⟦*Note:* There is nothing logically inconsistent about both clocks ticking slowly:

> Bob standing in Los Angeles thinks (correctly!) that Tokyo is below his feet, while Aiko standing in Tokyo thinks (correctly!) that Los Angeles is below her feet. This *would* be a logical contradiction if the Earth were flat, but the Earth is not flat.
>
> Ivan thinks (correctly!) that Veronica's clock ticks slowly, while Veronica thinks (correctly!) that Ivan's clock ticks slowly. This *would* be a logical contradiction if two events simultaneous in one frame were simultaneous in all frames, but they are not.

The difference between these two situations is that you're familiar with the first and unfamiliar with the second. Through these notes, you are becoming familiar with "this strange and beautiful Universe, our home."[1]⟧

[1] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973) page v.

1.2. *Length contraction.* Ivan says Veronica's rods are short, Veronica says Ivan's rods are short. This is not a logical contradiction because. . .

a. a moving clock ticks slowly.

b. Ivan's clocks tick slowly, so by distance = speed × time, the distance must be smaller too.

c. it takes some time for light to travel the length of the meter stick.

d. two events simultaneous in Ivan's frame may not be simultaneous in Veronica's.

1.3. *Time dilation.* A moving clock ticks slowly because. . .

a. time passes slowly in the moving frame.

b. the clock was damaged during acceleration.

c. the observer is looking at "old light" which required a finite time to get from the clock to the observer.

1.4. *How do two moving clocks fall out of sync?* A pair of clocks is initially synchronized. Each clock undergoes an identical acceleration program until both clocks are moving at constant speed $0.9c$. The two clocks fall out of synchronization because. . .

a. the rear clock has been moving for longer, so its reading falls behind that of the front clock.

b. the front clock has been moving for longer, so its reading falls behind that of the rear clock.

c. during the acceleration process, the phenomena of general relativity are in play ("gravitational time dilation").

1.5. *Interval.* Starting from the Lorentz transformation equations, show that the quantity defined in equation (1.3) is, as claimed, the same in all reference frames, i.e. that

$$(ct)^2 - [(x)^2 + (y)^2 + (z)^2] = (ct')^2 - [(x')^2 + (y')^2 + (z')^2]. \tag{1.6}$$

Thus for the common sense Galilean transformation, t is the same in all reference frames, while interval is not. For the correct Lorentz transformation, the opposite holds. ⟦*Clue:* This problem is nothing more than algebra, but algebra goes more smoothly when it's informed by physical insight. The variables t, V, and c fall naturally in two packets: V/c (dimensionless velocity as a fraction of light speed) and ct (time measured in meters). Don't rend these packets apart. (Some people find it convenient to work with the symbols $\beta = V/c$ and $x_0 = ct$ in place of t, V, and c, so that it's impossible to rend the packets apart!)⟧

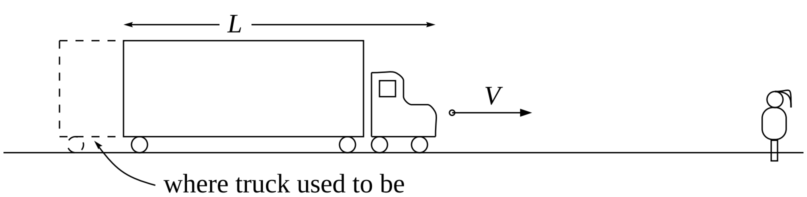
1.6. *Interval vs. distance.* In mathematics, the four properties of a distance function (or, to use the technical term, a *metric*) are non-negativity, symmetry, triangle inequality, and the property that the distance between two events is zero if and only if the two events are identical (sometimes called "the identity of indiscernibles"). Using this last property, show that the invariant interval between two events is *not* a distance function in this mathematical sense.

1.7. *Time dilation derivation.* Let T_0 represent the time ticked off by a clock. In frame F, that clock moves at speed V. Starting from the Lorentz transformation equations, show that the time elapsed in frame F while the moving clock ticks off time T_0 is

$$T = \frac{T_0}{\sqrt{1-(V/c)^2}}. \tag{1.7}$$

1.8. *Visual appearance of a moving truck.* Sometimes students pick up the misimpression that the effects of relativity are not real, but are appearance due to the finite speed of light.

For example, a truck of length L moves down the highway at speed V. Jessica, standing on the shoulder of the highway, watches the truck approach. Light from the nose of the truck takes some time to reach Jessica, but light from the tail of the truck takes even more time to reach her, because it has to travel further. Thus Jessica sees the nose of the truck as it was some time ago, but the tail of the truck as it was even more time ago, so the length she sees is longer than the true length of the truck.



This effect exists and is real, but it's not relativistic length contraction. When I say that a truck has length L_0 in its own reference frame and a shorter length L in the highway's reference frame, I mean that *really is* shorter, not just that it *appears* shorter due to the finite speed of light or some other effect. The visual-appearance *lengthening* effect cannot in any way explain relativistic length *contraction* — after all, it even goes the wrong direction!

Show that the visual-appearance length of the truck is

$$\frac{1}{1 - V/c} L.$$

1.9. *Galactic journey.* Veronica journeys from one edge of our galaxy to the other — 100,000 light years — while aging only 10 years. How fast was she traveling? (Present your answer to nine significant digits. *Clue:* There are many ways to solve this problem, but think about the invariant interval.)

1.10. *Flushing out an error.* Watch the music video "I Lost on Jeopardy" by "Weird Al" Yankovic, a parody of "[Our Love's in] Jeopardy" by the Greg Kihn Band:

http://www.youtube.com/watch?v=BvUZijEuNDQ

Find and correct the error in relativity.

Chapter 2

A Collision

2.1 Why we need relativistic dynamics

The first chapter dealt with the consequences of relativity for ideas about space and time. Are there consequences for things like force, momentum, and energy? Of course!

1. How does force affect motion?

Newton: A body starting from rest subject to constant force F will have velocity $v = (F/m)t$, which increases without bound when t increases.

Einstein: But v can't exceed c! Newton's formula, although an excellent approximation for small velocities, must be wrong.

Newton:

$$\vec{F}^{\text{net}} = m\vec{a} = \frac{d\vec{p}}{dt}.$$

Einstein: So you claim, but which t do you mean? Time as ticked off in the Earth's frame, in Mars's frame, in the space shuttle's frame, in the particle's frame?

2. What is the origin of force?

Newton: The gravitational force on the Earth due to the Sun is

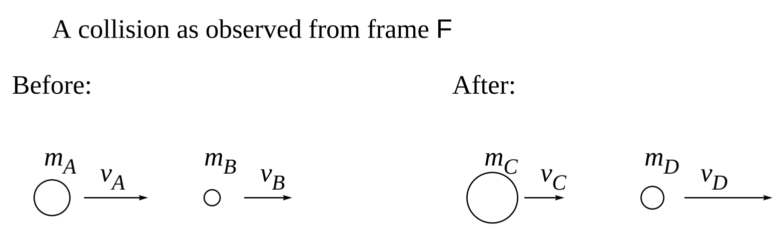
$$G\frac{m_E m_S}{r^2}.$$

Einstein: This formula says that if you move the Sun, the gravitational force on the Earth changes instantly! Relativity demands a time delay of about eight minutes — the time required for light to travel from the Sun to the Earth. Newton's formula, although an excellent approximation for small velocities, must be wrong.

Chapters 2 through 8 of these notes treat the "How does force affect motion?" question. The "What is the origin of force?" question is touched upon in chapter 9, but this is only an introduction. The pursuit of this question over the last century blossomed into the development of field theory, including Einstein's theory of general relativity (which is a relativistic field theory of gravity), the theory of quantum electrodynamics (QED), and the theory of quantum chromodynamics (QCD, concerning the nuclear strong force).

2.2 The situation

We begin our exploration of relativistic dynamics by seeking the proper — that is, the most useful —- definition of momentum in relativity. There are several ways to do this: if you're interested in history, you might want to look up how Einstein did it in his 1905 paper "On the electrodynamics of moving bodies" that founded relativity. Chapter 3 of this book, "Another Momentum Motivation" does it another way. But for now we begin by investigating a simple collision in one dimension.



Two bodies approach each other and interact, then two bodies draw away from each other. While the two bodies are close and interacting, they might be doing anything: There might be friction between them, in which case the classical kinetic energy would not be conserved. They might exchange atoms in which case the final mass m_C would not be the same as the initial mass m_A. The two bodies might stick together, in which case $v_C = v_D$. There might even be chemical reactions in which case the two final bodies would have different compositions from the two initial bodies. None of this matters: if there are no external forces, momentum ought to be conserved. We investigate this collision classically using the classical definition of momentum, and then with two different candidate definitions for relativistic momentum.

2.3 Classical analysis

First analyze this collision in frame F, the frame used in the sketch. The classical definition of momentum is $\vec{p} = m\vec{v}$, so momentum conservation says that

$$m_A v_A + m_B v_B = m_C v_C + m_D v_D. \tag{2.1}$$

What about an analysis in frame F'? Frames F and F' are equally good, so presumably momentum is conserved in both. In frame F' momentum conservation says that

$$m_A v'_A + m_B v'_B = m_C v'_C + m_D v'_D, \tag{2.2}$$

but we also know that

$$v'_A = v_A - V \tag{2.3}$$

so

$$m_A v_A + m_B v_B - (m_A + m_B)V = m_C v_C + m_D v_D - (m_C + m_D)V. \tag{2.4}$$

Subtracting the two momentum conservation equations (2.1) and (2.4) tells us that

$$(m_A + m_B)V = (m_C + m_D)V \tag{2.5}$$

and, because this applies to any frame velocity V, we find the frame-independent result

$$m_A + m_B = m_C + m_D, \tag{2.6}$$

the conservation of mass!

If momentum is conserved in all inertial reference frames, then mass must also be conserved. The conservation of mass is not an independent principle: it follows from the conservation of momentum plus the idea that any inertial reference frame is as good as any other inertial reference frame (the "principle of relativity").

2.4 Relativistic analysis, first candidate definition

The obvious idea for relativistic momentum is to use the same definition that worked so well for classical momentum, namely

$$\vec{p} = m\vec{v}. \tag{2.7}$$

We analyze the collision in first frame F, where momentum conservation says that

$$m_A v_A + m_B v_B = m_C v_C + m_D v_D. \tag{2.8}$$

A momentum conservation analysis in frame F' says that

$$m_A v'_A + m_B v'_B = m_C v'_C + m_D v'_D, \tag{2.9}$$

but we also know that

$$v'_A = \frac{v_A - V}{1 - v_A V/c^2} \tag{2.10}$$

so

$$m_A \frac{v_A - V}{1 - v_A V/c^2} + m_B \frac{v_B - V}{1 - v_B V/c^2} = m_C \frac{v_C - V}{1 - v_C V/c^2} + m_D \frac{v_D - V}{1 - v_D V/c^2}. \tag{2.11}$$

And now we're stuck. In this equation the quantities V *don't* just cancel out, so using this candidate definition conservation of momentum *does* depend on reference frame. The momentum given by this definition isn't conserved in all reference frames. We must either abandon momentum conservation in all frames, or else find a different definition for momentum.

2.5 Relativistic analysis, second candidate definition

Let's look again at the definition (2.7):

$$\vec{p} = m\vec{v} = m\frac{d\vec{r}}{dt}. \tag{2.12}$$

Is this really so obvious? When we take a derivative with respect to time, why is time in frame F so important? We're looking for a property associated with the particle as well as the frame, so why should we necessarily use the frame's time? Let's use the particle's own time, the so called[1] *proper time* τ.

Our new candidate definition of momentum is

$$\vec{p} = m\frac{d\vec{r}}{d\tau}. \tag{2.13}$$

A change in proper time τ is related to a change in frame F time t through the time dilation result (see equation 1.7)

$$dt = \frac{d\tau}{\sqrt{1 - (v/c)^2}} \tag{2.14}$$

[1] The word "proper" is irritating. Any inertial frame is as good as any other inertial frame, so why should the time in one of those frames be considered more "proper" than any other time? The word origin is that the "proper" in "proper time" derives not from the English "proper" meaning "respectable, genteel", but from the French "propre" meaning "own". The particle's "proper time" means the particle's "own time".

where v is the velocity of the particle in frame F. Hence our new candidate definition is that in frame F, where the particle moves with velocity $\vec{v}$, the momentum is

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-(v/c)^2}}. \tag{2.15}$$

How does this candidate definition apply to the collision we've already looked at? The momentum of particle A in frame F is

$$\frac{m_A v_A}{\sqrt{1-(v_A/c)^2}}. \tag{2.16}$$

The momentum of particle A in frame $\mathsf{F'}$ is

$$\frac{m_A v'_A}{\sqrt{1-(v'_A/c)^2}}, \tag{2.17}$$

which, after some algebra (see problem 2.3, *Necessary algebra*), is found to equal

$$\frac{m_A v_A}{\sqrt{1-(v_A/c)^2}}\frac{1}{\sqrt{1-(V/c)^2}} - \frac{m_A}{\sqrt{1-(v_A/c)^2}}\frac{V}{\sqrt{1-(V/c)^2}}. \tag{2.18}$$

The reasoning now is familiar from the classical case: Multiply the momentum conservation equation in frame F by $1/\sqrt{1-(V/c)^2}$ and subtract the momentum conservation equation in frame $\mathsf{F'}$. The result is a new conserved quantity: namely

$$\frac{m_A}{\sqrt{1-(v_A/c)^2}} + \frac{m_B}{\sqrt{1-(v_B/c)^2}} = \frac{m_C}{\sqrt{1-(v_C/c)^2}} + \frac{m_D}{\sqrt{1-(v_D/c)^2}}. \tag{2.19}$$

If this second candidate for momentum is conserved in all inertial reference frames, then the sum over all particles of $m/\sqrt{1-(v/c)^2}$ must also be conserved. The conservation of this quantity is not an independent principle: it follows from the conservation of momentum plus the idea that any inertial reference frame is as good as any other inertial reference frame (the "principle of relativity").

2.6 Another conserved quantity

What we have presented so far is motivation. It asks us to focus our experiments on the quantity

$$\frac{m\vec{v}}{\sqrt{1-(v/c)^2}}, \tag{2.20}$$

summed over all particles, and the quantity

$$\frac{m}{\sqrt{1-(v/c)^2}}, \tag{2.21}$$

summed over all particles. Our motivation suggests that both of these quantities will be conserved. Do experiments agree?

As you can imagine, experiments with relativistic particles are not easy to do, and it took a lot of effort to perform and interpret them. Many blind alleys were explored, many graduate students were harried. I'll summarize a long history: These quantities are indeed conserved.[2]

The question now is: How should we interpret the quantity $m/\sqrt{1-(v/c)^2}$ which is conserved and thus important? The relativistic momentum

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-(v/c)^2}} \tag{2.22}$$

is only a little different from the classical momentum $\vec{p} = m\vec{v}$. But this new quantity seems unlike anything we've ever seen before.

When $v = 0$ our new quantity is just the mass, which we expect to be conserved. But what is this quantity in the limit where v is much smaller than c, but not so small as to be considered zero? We approach this limit through a Taylor series. Perhaps you remember that the Taylor series for $(1+\epsilon)^n$ is

$$(1+\epsilon)^n = 1 + n\epsilon + \tfrac{1}{2}n(n-1)\epsilon^2 + \cdots. \tag{2.23}$$

(If you don't remember this, you should derive it and memorize it now. It's one of the most useful formulas you'll ever encounter.) Applied to the small quantity $\epsilon = -(v/c)^2$ with $n = -\frac{1}{2}$, it tells us that

$$\frac{1}{\sqrt{1-(v/c)^2}} \approx 1 + \tfrac{1}{2}(v/c)^2. \tag{2.24}$$

For historical reasons we focus on c^2 times our new quantity, namely

$$\frac{mc^2}{\sqrt{1-(v/c)^2}} \approx mc^2(1 + \tfrac{1}{2}(v/c)^2) = mc^2 + \tfrac{1}{2}mv^2. \tag{2.25}$$

Well, $\frac{1}{2}mv^2$! That's an old friend! The quantity we've come across is a relativistic generalization of kinetic energy! (Remember that in classical mechanics only *changes* in energy are physically significant: We can alter all the energies of a problem by any given sea-level shift, and the changes will be unaffected. The term mc^2 represents such a shift...for typical velocities, a shift very large compared to $\frac{1}{2}mv^2$, but nevertheless merely a classical shift.)

In short, we define the relativistic momentum of a particle by

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-(v/c)^2}} \tag{2.26}$$

[2]An outline of the experimental tests is presented in the Wikipedia article "Tests of relativistic energy and momentum". See also Mark P. Haugan and Clifford M. Will, "Modern tests of special relativity," *Physics Today*, **40** (May 1987) 69–76.

and the relativistic energy of a particle by

$$E = \frac{mc^2}{\sqrt{1-(v/c)^2}}. \tag{2.27}$$

Experiment shows that these quantities, summed over all particles in an isolated system, are conserved in all reference frames.

⟦*Warning:* Conservation means this: (a) Start with several particles far apart, and sum up these four quantities over all particles. (b) Allow the particles to come together and interact. The interactions might change a lot of things: they might make particles split into pieces, or might make several particles stick together. (c) When the resulting particles are far apart and no longer interacting, sum up these four quantities again. The four sums will be the same as they were in the beginning.

If you were to do the four sums while the particles were interacting, then in general you would *not* find them to be the same. Why?

Remember from your study of electromagnetism that fields as well as particles carry energy and momentum. To find the total energy and momentum at all instants, you must integrate over the field energy and momentum as well as sum over the particle energy and momentum. These notes consider situations where the particles start off so far apart that they aren't interacting, then come together (during a "collision"), and finally scatter so far apart that they aren't interacting any longer. Only when the particles are close and interacting will you have to consider field energy and momentum as well as particle energy and momentum. This is why we'll rarely treat potential energy in these notes: relativistically, potential energy is energy carried in fields.⟧

Problems

2.1. *A third candidate definition.* Before equation (2.13) we argued that in equation (2.12) we might use the particle's time τ instead of the frame's time t. We could continue in this vein and argue that for $\vec{r}$ we should use the particle's position in the particle's frame instead of the particle's position in the frame F. Show that this choice generates a sterile result.

2.2. *Logical inversion.* We reasoned that momentum was conserved in all inertial frames, and concluded (classically) that mass was conserved or (relativistically) that energy was conserved. Turn this reasoning around: Assume that momentum is conserved in frame F, and show that momentum is conserved in all frames provided (classically) that mass is conserved or (relativistically) that energy is conserved.

2.3. *Necessary algebra.* Show that if

$$v' = \frac{v - V}{1 - vV/c^2}, \tag{2.28}$$

then

$$\frac{v'}{\sqrt{1 - (v'/c)^2}} = \frac{v - V}{\sqrt{1 - (v/c)^2}\,\sqrt{1 - (V/c)^2}}. \tag{2.29}$$

2.4. *Relativistic energy and momentum, I.* A particle of mass m is given so much energy that its total relativistic energy is equal to three times its rest energy (that is, $E = 3mc^2$). Find its resulting speed (as an expression involving c) and momentum (as an expression involving mc). How do these results change if the total energy is six times its rest energy?

2.5. *Relativistic energy and momentum, II.* A particle of mass m has total relativistic energy equal to γ times its rest energy (that is, $E = \gamma mc^2$). What is its speed? Its momentum? Does its speed have the proper limit as $\gamma \to \infty$?

2.6. *Relativistic energy: a new proposal.* A friend tells you: "I have a new idea about relativistic energy. That old fogey Einstein got it all wrong! In fact, relativistic energy should be defined not as

$$E = \frac{mc^2}{\sqrt{1 - (v/c)^2}} \quad \text{but as} \quad E = \frac{mc^2}{\sqrt{1 - (v/c)^4}}."$$

Prove your friend wrong. (*Clue:* Examine the classical limit $v \ll c$ of this formula using the result that, when $|\epsilon| \ll 1$, $(1 + \epsilon)^n \approx 1 + n\epsilon$.)

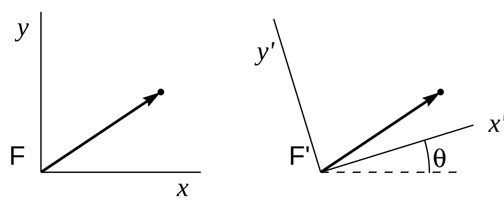
Chapter 3

Another Momentum Motivation

This chapter presents another way to motivate the definition of relativistic momentum presented in the previous chapter.

3.1 What is a vector?

Think of a vector pointing from the origin to a specific point in two-dimensional space.[1]



Frequently you'll hear people say that the vector is the same as the ordered pair (x, y), or that

$$\vec{r} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.1}$$

[1]This section concerns "vector" as defined by Gregorio Ricci-Curbastro and Tullio Levi-Civita in the year 1900. Other definitions of "vector" exist, including the more general and more abstract "vector space" definition by Giuseppe Peano in 1888, which was generalized by David Hilbert and Erhard Schmidt in 1908.

That's not exactly correct. What people should say is

$$\text{The vector } \vec{r} \text{ is represented by the column matrix } \begin{pmatrix} x \\ y \end{pmatrix} \text{ in the frame } \mathsf{F} \tag{3.2}$$

or

$$\text{The vector } \vec{r} \text{ has the coordinates } \begin{pmatrix} x \\ y \end{pmatrix} \text{ in frame } \mathsf{F}. \tag{3.3}$$

That's because in a different reference frame F′, the same vector is represented by a different column matrix:

$$\text{The vector } \vec{r} \text{ is represented by the column matrix } \begin{pmatrix} x' \\ y' \end{pmatrix} \text{ in the frame } \mathsf{F'}. \tag{3.4}$$

The sketch above shows the same vector, drawn once with the axes of frame F and then again in with the axes of frame F′. That one vector is represented by two different column matrices, which happen to be related through

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{3.5}$$

that is

$$\begin{aligned} x' &= +(\cos\theta)x + (\sin\theta)y \\ y' &= -(\sin\theta)x + (\cos\theta)y. \end{aligned} \tag{3.6}$$

For example, if $\theta = 45°$ then the vector represented by $(1, 1)$ in frame F is represented by $(\sqrt{2}, 0)$ in frame F′. If you said $\vec{r} = (1, 1)$ and $\vec{r} = (\sqrt{2}, 0)$, then it would follow that $(1, 1) = (\sqrt{2}, 0)$, which is obviously false! On the other hand the more accurate expressions like relation (3.2) are cumbersome, and difficult to pronounce. The new symbol $\doteq$ is used to mean the phrase "is represented by" so, in place of (3.1) or (3.2), we write

$$\vec{r} \doteq \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.7}$$

One way to express this idea is to say that the vector $\vec{r}$ has the name (x, y) in frame F and the different name (x', y') in frame F′. The vector is one thing but it has two different names, depending on which frame you use. In the same way a tree has the name "tree" in English and the different name "baum" in German. The tree is one thing but it has two different names, depending on which language you use.

While the coordinates of a vector depend on the frame, the *length* of a vector is *invariant*, that is, the same in all frames:

$$r^2 = (x)^2 + (y)^2 = (x')^2 + (y')^2. \tag{3.8}$$

There is a simple physical interpretation for r: it is the length that would be found by a tape measure laid along $\vec{r}$. The equation above shows how to calculate r from the coordinates, but you don't *need* a coordinate system to calculate r.

Because the distance r depends on the geometry of $\vec{r}$ alone, whereas the coordinates x and y depend on the geometry of $\vec{r}$ plus the choice of coordinate axes, you would expect that many, if not all, physical effects will depend on r rather than on, say, x'. If two particles are separated by distance r, it makes sense that the magnitude of the force between the two particles would depend upon r. It doesn't make sense that the magnitude of that force would depend upon the human choice of which coordinate system to use. (Just as a tree looks and behaves the same way whether you describe it as a "tree" or as a "baum".)

Similar results hold for three-dimensional vectors, except that if height z is measured in feet, while horizontal distances x and y are measured in miles, then the invariant length of the vector is

$$r^2 = [x]^2 + [y]^2 + [z/(5280 \text{ ft/mi})]^2. \tag{3.9}$$

3.2 What is a four-vector?

Just as the coordinates of an ordinary vector have a given transformation property (3.5) when transforming between two frames with relative axis rotation, so the coordinates of a *four-vector* have the Lorentz transformation property (1.1) when transforming between two frames in relative motion. For the case of ordinary vectors it made sense to covert from heights measured in feet to heights measured in miles, and for the case of four-vectors it makes sense to convert from times t measured in seconds to times ct measured in meters. (Make sure you understand that the quantity ct has the dimensions of length.) Using these quantities, the Lorentz transformations for the coordinates of an event are

$$\begin{aligned} ct' &= \frac{ct - (V/c)x}{\sqrt{1-(V/c)^2}} \\ x' &= \frac{x - (V/c)ct}{\sqrt{1-(V/c)^2}} \\ y' &= y \\ z' &= z \end{aligned} \tag{3.10}$$

We say

The four-vector $\mathbb{r}$ for an event is represented by the row matrix $[ct, x, y, z]$ in frame F (3.11)

or

$$\mathbb{r} \doteq [ct, x, y, z]. \tag{3.12}$$

The "invariance of interval" result (1.3) is that the combination of coordinates

$$(ct)^2 - (x^2 + y^2 + z^2) \tag{3.13}$$

is the same in all reference frames.

There is no simple physical "tape measure" interpretation of interval, in the way there was for the length of an ordinary position vector. Nevertheless it has a significance similar to the significance of length: the effect of one event on a second is likely to depend upon the interval separating those two events, and unlikely to depend upon the human choice of which reference frame to use.

3.3 Four-momentum

There are many different ordinary vectors: position $\vec{r}$, velocity $\vec{v}$, momentum $\vec{p}$, and so forth. They have in common that the coordinates of any vector transform under a rotation of axes in exactly the same way (3.5) that the coordinates representing the position of a point transform.

Similarly, there are many different four-vectors. They have in common that the coordinates of any four-vector transform under a Lorentz transformation in exactly the same way (3.10) that the coordinates representing the time and position of an event transform.

In Newtonian mechanics momentum is defined as

$$\vec{p} = m\frac{d\vec{r}}{dt}. \tag{3.14}$$

This is a vector because m is a scalar (the same regardless of rotation of axes) and t is a scalar.

How shall we define four-momentum in relativistic mechanics? We want something like

$$\mathbb{p} = m\frac{d\mathbb{r}}{dt_?}. \tag{3.15}$$

The quantity m is a four-scalar (the same regardless of Lorentz transformation). So far so good. But which time should we use for "$t_?$"? If we use time in, say, the Earth's frame, that time is *not* a four-scalar, because time *is* different from one frame to another. Before moving on, think about how you could select a time that is the same regardless of Lorentz transformation, i.e. a time that is a four-scalar.

If you use time in the Earth's frame, that's not a four-scalar, because there's nothing special about the Earth's frame. If you use time in a frame moving at $\frac{1}{2}c$ relative to the Earth, that's not a four-scalar, because there's nothing special about this frame, either. There's only one time that's special, and that's the time elapsed in the particle's own frame...the time ticked off by a wristwatch attached to the particle. This is called the *proper time* τ. Different frames disagree about what time it is, but all frames agree upon the time elapsed on the particle. The correct definition of four-momentum is

$$\mathbb{p} = m\frac{d\mathbb{r}}{d\tau}. \tag{3.16}$$

Thinking about proper time leads to the correct definition, but it's not the easiest way to conduct experiments. The electron in your laboratory probably isn't wearing a wristwatch! So while the above definition is the simplest conceptually, if you want to do experiments you'll want to write down a result using the coordinates that you measure in your lab frame. The lab frame time t and proper time τ are related through (see equation 1.7)

$$dt = \frac{d\tau}{\sqrt{1-(v/c)^2}} \tag{3.17}$$

where $d\tau$ is the time elapsed on the particle, dt is the time elapsed in the laboratory, and v is the speed of the particle in the laboratory.

So, in some particular inertial frame

$$\mathbb{r} \doteq [ct, x, y, z] \tag{3.18}$$

and

$$\mathbb{p} = m\frac{d\mathbb{r}}{d\tau} = m\frac{dt}{d\tau}\frac{d\mathbb{r}}{dt} \doteq \frac{m}{\sqrt{1-(v/c)^2}}[c, v_x, v_y, v_z]. \tag{3.19}$$

The last three components of this four-vector are easy to interpret: They say that the relativistic momentum in a particular frame is defined as

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-(v/c)^2}}. \tag{3.20}$$

The initial component

$$\frac{mc}{\sqrt{1-(v/c)^2}} \tag{3.21}$$

is the tough one. But we've already interpreted it back at section 2.6, "Another conserved quantity": this is proportional to relativistic energy. The four-momentum is

$$\mathbb{p} \doteq \frac{m}{\sqrt{1-(v/c)^2}}[c, v_x, v_y, v_z] = [E/c, p_x, p_y, p_z]. \tag{3.22}$$

If the total momentum is to be conserved in *all* inertial frames, then it's not enough for p_x, p_y, and p_z to be conserved. Because E/c mixes up with p_x through the Lorentz transformation, if p_x is conserved in all frames then E must be conserved in all frames, too.

The same argument that proves the interval of an event

$$(ct)^2 - (x^2 + y^2 + z^2) \tag{3.23}$$

to be invariant — the same in all reference frames — also proves the quantity

$$(E/c)^2 - (p_x^2 + p_y^2 + p_z^2) = (E/c)^2 - p^2 \tag{3.24}$$

to be invariant. Because it's the same in all frames, it's the same in the particle's own frame, where $p = 0$ and $E = mc^2$. This is more conveniently written after multiplying through by c^2. The quantity

$$E^2 - (pc)^2 = (mc^2)^2 \tag{3.25}$$

is the same in all reference frames.

3.4 Old style

Before about 1990, it was popular to define the "relativistic mass"

$$m_R \equiv \frac{m}{\sqrt{1 - (v/c)^2}}$$

so that

$$\vec{p} = m_R\vec{v} \quad \text{and} \quad E = m_R c^2.$$

(You may have see that last equation before.) This made some equations easier to remember, but others harder to remember. It had the disadvantages that (a) relativistic mass was not a four-scalar and (b) this mass was not the mass that enters into $\vec{F}^{\text{net}} = m\vec{a}$. In fact, in this old style one had to define not only a "relativistic mass" related to momentum, but also a "longitudinal mass" and a "transverse mass" related to force (see page 54 and problems 7.2 and 7.6). The regular old ordinary mass was called "rest mass" or "proper mass". This scheme has a lot of disadvantages[2] and is no longer used. I mention it only because you might look into some old book that used this old style.

[2]Here's one: The "relativistic mass" of any particle increases without bound depending upon reference frame. Thus every particle is a black hole when observed from a reference frame moving near light speed past that particle!

3.5 Summary

Time and space

The time and location of an event is specified by the time-space four-vector $\mathbb{r}$ whose components $[ct, x, y, z]$ transform from one frame to another through

$$\begin{aligned} ct' &= \frac{ct - (V/c)x}{\sqrt{1-(V/c)^2}} \\ x' &= \frac{x - (V/c)ct}{\sqrt{1-(V/c)^2}} \\ y' &= y \\ z' &= z. \end{aligned} \tag{3.26}$$

The combination of coordinates

$$(ct)^2 - x^2 - y^2 - z^2 \tag{3.27}$$

is *invariant*, the same in all reference frames.

Energy and momentum

The energy and momentum of a particle is specified by the energy-momentum four-vector

$$\mathbb{p} = m\frac{d\mathbb{r}}{d\tau} \doteq \frac{m}{\sqrt{1-(v/c)^2}}[c, v_x, v_y, v_z] = [E/c, p_x, p_y, p_z]. \tag{3.28}$$

whose components $[E/c, p_x, p_y, p_z]$ transform from one frame to another through

$$\begin{aligned} E'/c &= \frac{(E/c) - (V/c)p_x}{\sqrt{1-(V/c)^2}} \\ p'_x &= \frac{p_x - (V/c)(E/c)}{\sqrt{1-(V/c)^2}} \\ p'_y &= p_y \\ p'_z &= p_z. \end{aligned} \tag{3.29}$$

The combination of coordinates

$$E^2 - (pc)^2 = (mc^2)^2 \tag{3.30}$$

is *invariant*, the same in all reference frames.

In classical mechanics, for an isolated system total momentum is conserved and the total kinetic energy might or might not be conserved.

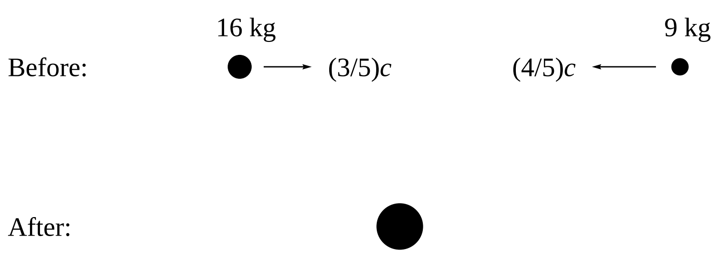
In relativistic mechanics, for an isolated system the total momentum and the total energy are always conserved (furthermore it's conserved in all intertial frames). [Although "total momentum" and "total energy" require not just summing over particles, but integrating over fields.]

Chapter 4

A Sticky Collision

4.1 A completely inelastic collision

Let's apply the conservation of energy and momentum to a specific case. A ball of bubble gum with mass 16 kg, and another ball of bubble gum with mass 9 kg, speed toward each other as shown:



The two balls stick together.

Before the collision, the total (horizontal) momentum and the total energy are given through

$$p^{\text{total}} = \sum_i \frac{m_i v_i}{\sqrt{1-(v_i/c)^2}} = \frac{(16\text{ kg})(\frac{3}{5}c)}{\frac{4}{5}} + \frac{(9\text{ kg})(-\frac{4}{5}c)}{\frac{3}{5}} = (12\text{ kg})c - (12\text{ kg})c = 0 \tag{4.1}$$

and

$$E^{\text{total}} = \sum_i \frac{m_i c^2}{\sqrt{1-(v_i/c)^2}} = \frac{(16\text{ kg})c^2}{\frac{4}{5}} + \frac{(9\text{ kg})c^2}{\frac{3}{5}} = (35\text{ kg})c^2. \tag{4.2}$$

After the collision, the single ball has momentum zero — it's at rest — and energy $(35\text{ kg})c^2$. Thus the mass of the single ball is 35 kg.

What? Two balls, of mass 16 kg and mass 9 kg, stick together and form a ball, not of mass 25 kg, but of mass 35 kg? Can that really be?

Yes. In relativity:

total momentum is the sum of the momenta of the constituents
and
total energy is the sum of the energies of the constituents
but
total mass is *not* the sum of the masses of the constituents.

4.2 Mass in relativity

Is the mass of a composite object equal to the sum of the masses of its constituents? The answer "yes" seems so natural and obvious that the question hardly needs asking. Yet relativity claims that the correct answer is "no"! (Instead, the energy of the composite is equal to the sum of the energies of its constituents.) As always, the test of correctness is experiment, not obviousness.

The masses of atoms and subatomic particles have been measured to very high accuracy (primarily through the technique of "mass spectroscopy"). For example, the mass of the proton is known to 11 significant digits. In these notes, I'll present only a handful of the many measurements available, and I'll round them down to seven decimal places, which is more than enough accuracy to prove my point. The masses here are given not in terms of the kilogram (abbreviated as "kg") but in terms of the "atomic mass unit" (abbreviated as "u"), which is about the mass of a hydrogen atom and exactly 1/12 the mass of a neutral unbound ground-state carbon-12 atom ($^{12}_{6}$C). (These data come from the National Institute of Standards and Technology through http://physics.nist.gov/cuu/Constants/index.html and from the Atomic Mass Data Center in Lanzhou, China, through http://amdc.impcas.ac.cn/evaluation/data2012/ame.html.) These sources give the mass values:

mass of electron	0.000 548 6 u
mass of proton	1.007 276 5 u
mass of $^{1}_{1}$H	1.007 825 0 u
mass of neutron	1.008 664 9 u
mass of $^{4}_{2}$He	4.002 603 3 u
mass of $^{8}_{4}$Be	8.005 305 1 u
mass of $^{26}_{14}$Si	25.992 333 8 u

So, does the mass of an atom equal the sum of the masses of its constituents? A $^4_2$He atom consists of two electrons, two protons, and two neutrons:

sum of masses of constituents	4.032 980 0 u
mass of $^4_2$He	4.002 603 3 u

No! The atom is less massive than the sum of its constituents!

Problem: Compare the masses of the following systems, each of which has the same constituents: (a) four electrons, four protons, and four neutrons (b) two $^4_2$He atoms, and (c) one $^8_4$Be atom.

four times mass of (electron plus proton plus neutron)	8.065 960 0 u
mass of $^8_4$Be	8.005 305 1 u
twice mass of $^4_2$He	8.005 206 6 u

That is, if you consider the $^8_4$Be atom as being made up of four electrons, four protons, and four neutrons, then its mass is *less* than the sum of the masses of its constituents. But if you consider the $^8_4$Be atom as being made up of two $^4_2$He atoms, then its mass is *more* than the sum of the masses of its constituents.

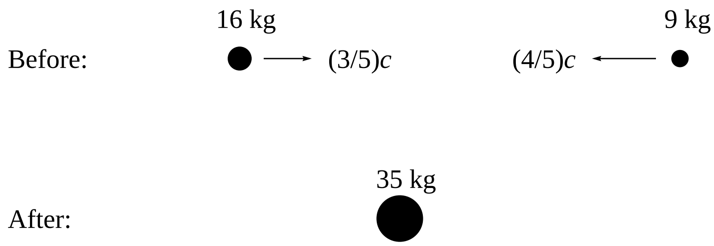
Problem: The molecule acetylene, $H-C\equiv C-H$, consists of 14 electrons, 14 protons, and 12 neutrons (provided that it's made from the most abundant isotopes of carbon and hydrogen, namely $^{12}_6$C and $^1_1$H). The atom silicon-26 ($^{26}_{14}$Si) has exactly the same constituents.

sum of masses of 14 electrons, 14 protons, and 12 neutrons	26.213 530 2 u
mass of acetylene molecule	26.015 650 0 u
mass of $^{26}_{14}$Si	25.992 333 8 u

Problems

4.1. *Sticky particles.* A putty ball of mass 5 kg is hurled at $v = \frac{12}{13}c$ toward a stationary putty ball of mass 2 kg. The two balls stick together. What is the mass and the speed of the resulting lump of putty? (*Clues:* $\sqrt{1-(\frac{12}{13})^2} = \frac{5}{13}$. In contrast to the situation in section 4.1, in this problem the resulting lump will *not* be at rest in the laboratory.)

4.2. *A sticky situation, reanalyzed.* Section 4.1 analyzed the following collision, in which two balls of bubble gum stick together:



This problem analyzes the same collision from the frame in which the 16 kg ball is at rest.

a. What is the velocity of the 9 kg ball in this frame?

b. What is the total momentum of the system in this frame?

c. What is the total energy of the system in this frame?

d. What is the velocity of the resulting glob in this frame?

e. What is the mass of the resulting glob (in any frame)?

4.3. *Sticky particles and the classical limit.* A putty ball moving at speed v collides with an identical stationary putty ball. The two balls stick together.

a. In classical mechanics, what is the speed of the resulting composite?

b. In relativistic mechanics, what is the speed of the resulting composite?

c. Does your result in part (b) have the proper limit when $v \ll c$?

d. Is the relativistic resulting speed greater than or less than the classical resulting speed?

e. Each of the two initial putty balls have mass m. What is the mass of the resulting composite?

f. Does your result in part (e) have the proper limit when $v \ll c$?

g. Is the relativistic resulting mass greater than or less than the classical resulting mass?

4.4. *Two-particle system.* Two particles move on the x-axis. Particle A has mass m_A and velocity (relative to frame F) v_A, particle B has mass m_B and velocity (relative to frame F) v_B.

a. Show that the two-particle system has mass M where

$$M^2 = m_A^2 + m_B^2 + 2m_A m_B \frac{1 - v_A v_B/c^2}{\sqrt{(1-(v_A/c)^2)(1-(v_B/c)^2)}}. \tag{4.3}$$

Frame F' moves relative to frame F at velocity V, so in this frame the two particles have velocities

$$v'_A = \frac{v_A - V}{1 - v_A V/c^2} \quad \text{and} \quad v'_B = \frac{v_B - V}{1 - v_B V/c^2}. \tag{4.4}$$

b. Show that in frame F', the system has the same mass M given above.

Chapter 5

Momentum, Energy, and Mass

Momentum in relativity differs a little from momentum in classical mechanics:

$$\begin{aligned} \text{for a particle,} \quad & \vec{p} = \frac{m\vec{v}}{\sqrt{1-(v/c)^2}} \\ \text{for a system,} \quad & \vec{p}^{\,\text{total}} = \sum_i \vec{p}_i \\ \text{if system has no external forces,} \quad & \vec{p}^{\,\text{total}} \text{ is conserved} \end{aligned}$$

Energy differs quite a bit:

$$\begin{aligned} \text{for a particle,} \quad & E = \frac{mc^2}{\sqrt{1-(v/c)^2}} \\ \text{for a system,} \quad & E^{\text{total}} = \sum_i E_i \\ \text{if system has no external forces,} \quad & E^{\text{total}} \text{ is conserved} \end{aligned}$$

(In particular, that last line is *not* true in classical mechanics.) But mass differs most of all:

$$\begin{aligned} \text{for a particle,} \quad & E^2 - (\vec{p}c)^2 = (mc^2)^2 \\ \text{for a system, we } \textit{define} \quad & (E^{\text{total}})^2 - (\vec{p}^{\,\text{total}}c)^2 \equiv (M^{\text{total}}c^2)^2 \\ \text{if system has no external forces,} \quad & M^{\text{total}} \text{ is conserved} \\ \text{and with this definition,} \quad & M^{\text{total}} \neq \sum_i m_i \end{aligned}$$

This is the only reasonable definition of M^{total} (for example, it is the only sensible way to make M^{total} a four-scalar, the same in all reference frames) but it certainly results in new and unexpected properties for mass.

5.1 "Converting mass into energy"

The easiest way to interpret these new properties is to transform into a reference frame in which the system's total momentum is zero. In this so-called zero-momentum frame,

$$E^{\text{total}} = M^{\text{total}}c^2, \tag{5.1}$$

so increasing the *energy* of the system results in increasing the *mass* of the system.

Think again about the bubble gum collision of section 4.1: The initial wads had masses 16 kg and 9 kg, the final wad has mass 35 kg. Although we didn't mention it at the time, it also has high temperature: We know from classical experience that in an inelastic collision kinetic energy isn't conserved, it's converted into thermal energy. The increased thermal energy of the wad is reflected in E^{total}, which in turn is reflected in M^{total} through equation (5.1). We could have gotten to this final condition through a different route: We could have stuck the two wads together to form a 25 kg wad, then heated that wad with a blowtorch to give it enough thermal energy, and the increase in thermal energy would have then increased the wad's mass to 35 kg.

Another situation: A battery connects to a resistor, the whole circuit isolated from outside forces. Because of this isolation, M^{total} is conserved. Current flows, the resistor warms and thus, as already seen, the mass of the resistor increases. How can M^{total} remain constant? The mass of the battery *must* decrease as it drains: the fresh (high energy) battery has more mass than the drained (low energy) battery.

Examples like this can be repeated without limit: The fact that the mass of the system in any frame is proportional to the energy of the system in the zero-momentum frame convinces us that *any* increase in energy has to result in an increase in mass.

A bottle of gas has more mass when hot than when cold.
A spring has more mass when compressed (or when stretched) than when relaxed.
A capacitor has more mass when charged than when discharged.
A battery has more mass when fresh than when drained.
An atom has more mass when excited than when in the ground state.
A nucleus has more mass when excited than when in the ground state.

Our reasoning assures us that all of these statements are true. But c^2 is so large that the change in mass is very small, and as a consequence experiment has directly verified only the last of these statements.[1]

[1] For the most recent and most accurate of many tests, see S. Rainville, J.K. Thompson, E.G. Myers, J.M. Brown, M.S. Dewey, E.G. Kessler, R.D. Deslattes, H.G. Börner, M. Jentschel, P. Mutti, and D.E. Pritchard, "A direct test of $E = mc^2$," *Nature*, **438** (22 December 2005) 1096–1097.

⟦This discussion has skirted around potential energy (that is, field energy). In fact, the argument holds for any kind of energy, so the result must hold not only for kinetic energy located within the system but also for potential energy localized within the system. The examples in section 4.2 often had total masses *less* than the sum of the masses of the constituents, because the potential energy of, say, a $^4_2$He atom is *less* than the potential energy of two electrons, two protons, and two neutrons, all well separated.

More thought is needed for cases where the potential energy is not localized.[2] For example, you might think that a brick on the top floor of a skyscraper will have greater mass than that same brick on the ground floor of the skyscraper, because the higher brick has greater gravitational potential energy. The situation is more complex than this, however, because the gravitational potential energy is not localized within the brick; it belongs to the system of brick plus earth.⟧

You might object that this was just a definition of M^{total}, devoid of experimental consequences. No. Section 8.2 will show two ways in which the mass of a localized system is experimentally accessible.

Given that the mass of a system is *not* the sum of the masses of its constituents, how *are* these two quantities related? There's no simple general result, but in the zero-momentum frame (for situations with no field energy) there's a straightforward one. Define the kinetic energy of a particle as its energy above and beyond the rest energy:

$$E = mc^2 + \text{KE}. \tag{5.2}$$

Then, in the zero-momentum frame,

$$M^{\text{total}} = \sum_i m_i + \frac{1}{c^2}\sum_i \text{KE}_i. \tag{5.3}$$

(Note that M^{total} is the same in all frames, and $\sum_i m_i$ is the same in all frames, but $\sum_i \text{KE}_i$ is different in different frames. This equation holds only if the kinetic energies are taken in the zero-momentum frame.)

This equation, too, sheds light on the "sticky collision" of section 4.1. Each individual atom in the resulting blob has the same mass as it did before the collision, so $\sum_i m_i$ is the same before and after the collision. But $\sum_i \text{KE}_i$ is considerably larger after the collision than it was before, and this kinetic energy increase accounts for the 10 kg of "excess mass".

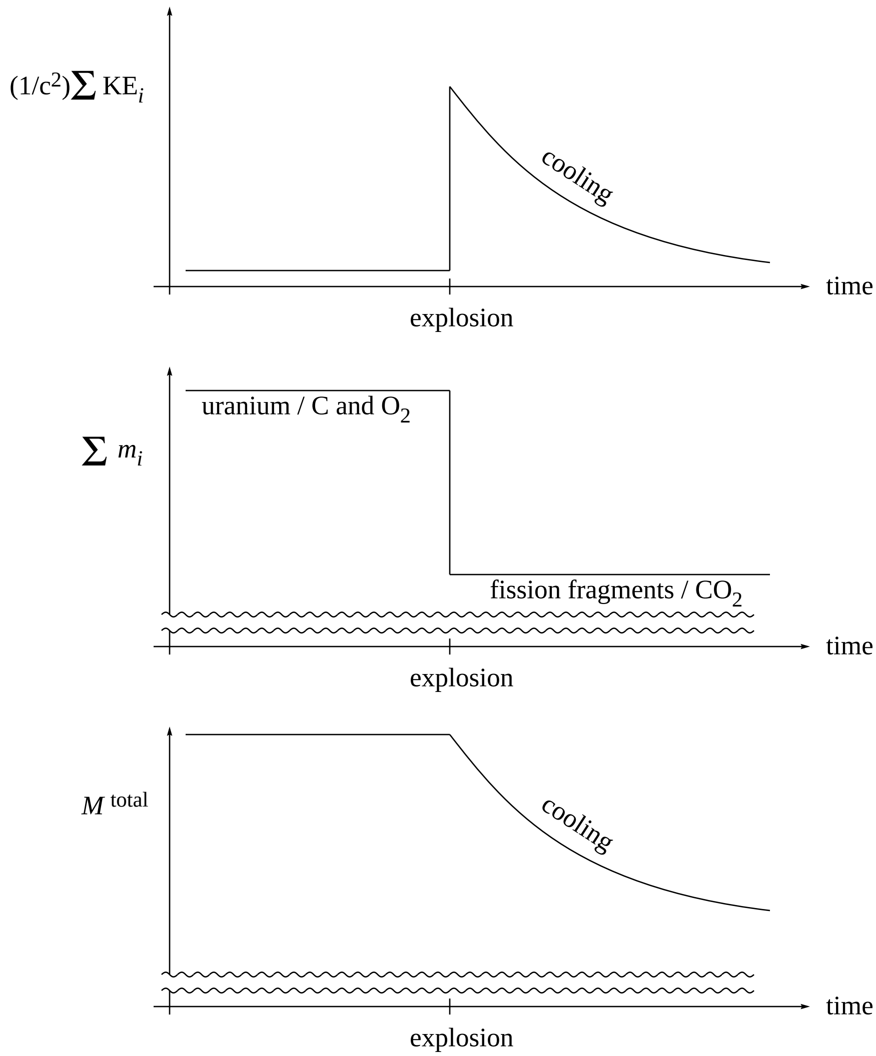
Sometimes you hear people say "a nuclear bomb converts mass into energy". What could this possibly mean? The quantities E^{total} and M^{total} are conserved (for

[2] "It is notoriously dangerous to speak of the momentum (or energy) of a configuration that is not localized in space." David Babson, Stephen Reynolds, Robin Bjorkquist, and David J. Griffiths, "Hidden momentum, field momentum, and electromagnetic impulse," *American Journal of Physics*, **77** (September 2009) 826–833.

an isolated system), so there's no question of changing either of them. Furthermore, mass is a four-scalar (the same in all reference frames), whereas energy is the time component of a four-vector (different from one reference frame to another). So these people can't be talking about something frame-independent. Instead, they're talking[3] about equation (5.3). When the nuclear bomb goes off, uranium nuclei fission into fragments. If you add up the mass of each fragment, you'll find a result *less* than the mass of the uranium nucleus. When the bomb explodes, $\sum_i m_i$ decreases but M^{total} does not change at all! This can happen only through an increase in $(1/c^2)\sum_i \text{KE}_i$.

After the bomb goes off, the thermal energy represented by $\sum_i \text{KE}_i$ seeps off into the environment, so M^{total} decreases as well. This does not violate any conservation law because the system is no longer isolated.

[3]Some treatments (for example, the video "The Real Meaning of $E = mc^2$" by "Space Time | PBS Digital Studios | Gabe Perez-Giz") emphasize equation (5.3), without ever pointing out that this relation holds only in the zero-momentum frame.



Finally, I emphasize that there's nothing qualitatively different between what a nuclear bomb does and what a chemical bomb does and what a match does. In a burning match carbon combines with oxygen to produce carbon dioxide. The mass of a CO_2 molecule is very slightly less than the mass of a C atom plus the mass of an O_2 molecule. Thus the product CO_2 must have increased kinetic energy. The nuclear bomb and the match both "convert mass into energy" in exactly the same sense. They differ only in the scale of conversion from $\sum_i m_i$ to $(1/c^2)\sum_i \mathrm{KE}_i$.

5.2 Massless particles

The formulas for $\vec{p}$ and E in terms of m and v are of course different from the familiar classical formulas. It's tempting to immediately rush in and use those new formulas. Tempting but a bad idea. As a fact of experimental life, it's hard to measure the velocity of a proton, but relatively easy to measure its energy or momentum. So instead of using results relating to velocity, its better to use expressions in terms energy and momentum. These are related through

$$E^2 - (pc)^2 = (mc^2)^2. \tag{5.4}$$

If you do need to know the velocity, and have mostly expressions involving energy and momentum, you can find it through

$$\frac{\vec{v}}{c} = \frac{\vec{p}c}{E}. \tag{5.5}$$

In fact, these two expressions are logically equivalent to

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-(v/c)^2}} \tag{5.6}$$

and

$$E = \frac{mc^2}{\sqrt{1-(v/c)^2}}. \tag{5.7}$$

That is, from equations (5.4) and (5.5) you can derive equations (5.6) and (5.7), or you can go in the other direction.

But the energy/momentum relations (5.4) and (5.5) are not just easier to use than the mass/velocity relations (5.6) and (5.7), they also open the door to a new possibility, a possibility undreamed of in classical mechanics, the possibility of a particle with zero mass.

In classical physics, if a particle has $m = 0$, then it has $p = mv = 0$. And if a particle has no momentum (also no energy) then it doesn't exist at all. In relativistic physics, equation (5.6) says *almost* the same thing: if $m = 0$, then in most cases $p = 0$. But there's one out: if $m = 0$ and $v = c$, then equation (5.6) gives $0/0$. To interpret this indeterminate form, turn to equation (5.4). We can indeed have a particle with $m = 0$, in which case $E = pc$. Equation (5.5) confirms that such a particle must have $v = c$.

Massless particles can exist, they can have energy, they can have momentum, but they can't travel at any speed except c.

A photon, a "particle of light", is a massless particle. It's been postulated that the "graviton" is a massless particle, but they've never been detected. Before 2001 it was thought that neutrinos were massless, but it's now well-established[4] that they have a

[4]Nathalie Palanque-Delabrouille *et al.*, "Neutrino masses and cosmology with Lyman-alpha forest power spectrum," *Journal of Cosmology and Astroparticle Physics* 6 November 2015.

mass with mc^2 less than about 0.1 eV. (In contrast, an electron has $mc^2 = 511\,000$ eV. If you are unfamiliar with the electron volt (eV) as a unit of energy, see problem 5.3, *Energy measures*.)

5.3 Transformation of space-time, of momentum-energy

Space and time transform according to

$$\begin{aligned} ct' &= \frac{ct - (V/c)x}{\sqrt{1-(V/c)^2}} \\ x' &= \frac{x - (V/c)ct}{\sqrt{1-(V/c)^2}} \\ y' &= y \\ z' &= z \end{aligned} \tag{5.8}$$

(note that ct has the dimensions of length). This is the *meaning* of the statement

$$\text{“}\mathbb{r} \doteq [ct, x, y, z] \text{ is a four-vector.”} \tag{5.9}$$

A consequence is that interval

$$(ct)^2 - (x^2 + y^2 + z^2) \tag{5.10}$$

is the same in all reference frames.

Momentum and energy transform according to

$$\begin{aligned} E'/c &= \frac{E/c - (V/c)p_x}{\sqrt{1-(V/c)^2}} \\ p'_x &= \frac{p_x - (V/c)(E/c)}{\sqrt{1-(V/c)^2}} \\ p'_y &= p_y \\ p'_z &= p_z \end{aligned} \tag{5.11}$$

(note that E/c has the dimensions of momentum). This is the *meaning* of the statement

$$\text{“}\mathbb{p} \doteq [E/c, p_x, p_y, p_z] \text{ is a four-vector.”} \tag{5.12}$$

A consequence is that the combination

$$(E/c)^2 - (p_x^2 + p_y^2 + p_z^2) \tag{5.13}$$

is the same in all reference frames.

5.4 Summary of energy, momentum, and mass in relativity

For a massive particle, the four-momentum $\mathbb{r}$ is

$$\mathbb{p} = m\frac{d\mathbb{r}}{d\tau} \doteq \frac{m}{\sqrt{1-(v/c)^2}}[c, v_x, v_y, v_z] \equiv [E/c, p_x, p_y, p_z]. \tag{5.14}$$

Consequences are:

$$E = \frac{mc^2}{\sqrt{1-(v/c)^2}} \tag{5.15}$$

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-(v/c)^2}} \tag{5.16}$$

$$E^2 - (pc)^2 = (mc^2)^2 \tag{5.17}$$

$$\frac{\vec{v}}{c} = \frac{\vec{p}c}{E} \tag{5.18}$$

The last two equations hold for massless as well as massive particles.

For a system of particles (no external interactions, no energy or momentum in fields):

$$E^{\text{total}} = \sum_i E_i \tag{5.19}$$

$$\vec{p}^{\ \text{total}} = \sum_i \vec{p}_i \tag{5.20}$$

$$E^{\text{total}} \text{ and } \vec{p}^{\ \text{total}} \quad \text{are conserved} \tag{5.21}$$

$$(E^{\text{total}})^2 - (\vec{p}^{\ \text{total}}c)^2 \equiv (M^{\text{total}}c^2)^2 \tag{5.22}$$

$$M^{\text{total}} \quad \text{is also conserved but}$$

$$M^{\text{total}} \neq \sum_i m_i \tag{5.23}$$

For an isolated system, the quantity in equation (5.22) is called "the conserved invariant".

Problems

5.1. *What is total mass?* In the bubble gum collision of section 4.1, what was the total mass of the system *before* the collision?

5.2. *"Converting energy into mass."* We've talked about the true meaning of the phrase "convert mass into energy". Is there ever a situation through which, in the same sense, "energy is converted into mass"? (*Clue:* See section 4.1.)

5.3. *Energy measures.* Atoms are small on a human scale, and consequently typical atomic energies are small on a human scale. For example, the energy required to strip the electron from a hydrogen atom is 2.18×10^{-18} J. If we were to measure typical atomic energies in joules, we'd run around all the time saying "ten to the negative eighteen", which is a real mouthful. Physicists avoid this mouthful by instead measuring atomic energies in the unit of "electron volts" (symbol eV). Despite its name, the electron volt is a unit of energy, not voltage. It is defined as the kinetic energy gained by an electron as it moves from a point where the electric potential is 0 volt to a point where the electric potential is 1 volt.

a. What is the eV in joules?

Chemists use a different, and equally legitimate, way of avoiding the mouthful. They don't consider energies for a single atom, but instead energies for a mole of atoms.

b. An energy of exactly 1 eV/atom corresponds to how many kJ/mol?

I like to remember this result as

$$1\text{eV/atom} \approx 100 \text{ kJ/mol}.$$

Thus a typical thermal energy at room temperature, $k_B T = \frac{1}{40}$ eV, is about the same as 2.5 kJ/mol. In contrast, the standard heat of formation of liquid water is −286 kJ/mol.

5.4. *Sticky particles, II.* Problem 4.1, *Sticky particles*, was:

> A putty ball of mass 5 kg is hurled at $v = \frac{12}{13}c$ toward a stationary putty ball of mass 2 kg. The two balls stick together. What is the mass and speed of the resulting lump of putty?

Solve this problem using the conserved invariant.

5.5. *X-rays.* In the lab frame, an X-ray photon travels right with energy 4.68 keV. In a frame traveling right at speed $V = \frac{3}{5}c$ relative to the lab, what is that photon's energy?

5.6. *Photon energy.* A photon has energy E_γ in the laboratory frame. What is its energy in a frame that runs after that photon with speed V (relative to the laboratory)? (*Moral of the story:* If you run after an electron — at speed V less than the electron's speed — then in your frame the electron has less speed and less energy than it has in the lab frame. But if you run after a photon, then in your frame the photon has the same speed and less energy than it has in the lab frame.)

5.7. *Two photons.* A photon of energy E_1 travels east, and a photon of energy E_2 travels west. Each photon, of course, has zero mass.

a. What is the total mass of the two-photon system?

b. The second photon reflects from a mirror so that both photons travel east. Now what is the total mass of the two-photon system?

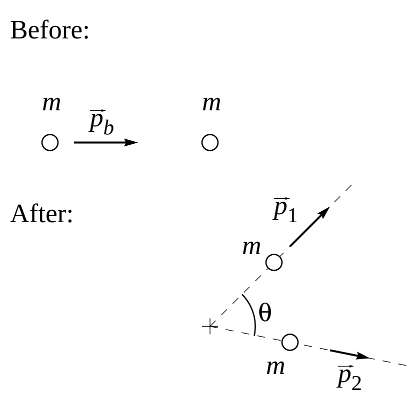
Chapter 6

Colliding Protons

To become familiar with these ideas concerning momentum, energy, and mass, we apply them to a specific situation, namely the collision of two protons.

6.1 Classical colliding protons

We begin with the non-relativistic, elastic collision of two particles with equal mass. Remember that in the non-relativistic context, "elastic" means that kinetic energy is conserved.



According to conservation of momentum

$$\vec{p}_b = \vec{p}_1 + \vec{p}_2 \tag{6.1}$$

while according to conservation of kinetic energy

$$\frac{p_b^2}{2m} = \frac{p_1^2}{2m} + \frac{p_2^2}{2m}. \tag{6.2}$$

(You might be more used to writing kinetic energy as $\frac{1}{2}mv^2$ rather than as $p^2/2m$. You can work this problem primarily through using momentum or primarily through using velocity. If you try using both you are likely to get confused with all the variables floating around, so I am going to write down only momentum relations.)

How should we begin to use these equations? You might be tempted to put the momentum conservation equation (6.1) into component form, but I'm going to adhere to my previous clue (page 10) "Don't rend natural packets apart" and keep the vector momentum as one packet.

From the momentum conservation equation we conclude that

$$p_b^2 = \vec{p}_b \cdot \vec{p}_b = (\vec{p}_1 + \vec{p}_2) \cdot (\vec{p}_1 + \vec{p}_2) = p_1^2 + 2\vec{p}_1 \cdot \vec{p}_2 + p_2^2, \tag{6.3}$$

while from the energy conservation equation we conclude that

$$p_b^2 = p_1^2 + p_2^2. \tag{6.4}$$

Comparing these two shows that

$$\vec{p}_1 \cdot \vec{p}_2 = |\vec{p}_1||\vec{p}_2| \cos\theta = 0. \tag{6.5}$$

There are three possible ways for this result to hold:

> Either $\vec{p}_2 = 0$ (the projectile particle misses the target particle)
> or $\vec{p}_1 = 0$ (dead center collision. . . projectile stops dead and target moves off with same velocity as the projectile had)
> or $\theta = 90°$.

If two particles move away from the collision, one will move away perpendicular to the other!

If you have access to a billiard table or an air hockey table, you should test this result experimentally. Remember that billiards and pucks are not perfectly elastic, and that there is kinetic energy in rotation as well as translation, so the 90° rule will not be obeyed *perfectly*. But it will be obeyed to remarkable precision.

It is quite characteristic that conservation of energy and momentum doesn't tell us exactly what happens, but instead leaves us with several possibilities. To describe the outcome of a two-dimensional collision we need four numbers ($p_{x,1}$, $p_{y,1}$, $p_{x,2}$, $p_{y,2}$). The conservation laws give us only three equations (x-momentum, y-momentum, energy). There are not enough equations to determine all four unknowns. The conservation laws do, however, rule out some possibilities.

6.2 Relativistic colliding protons

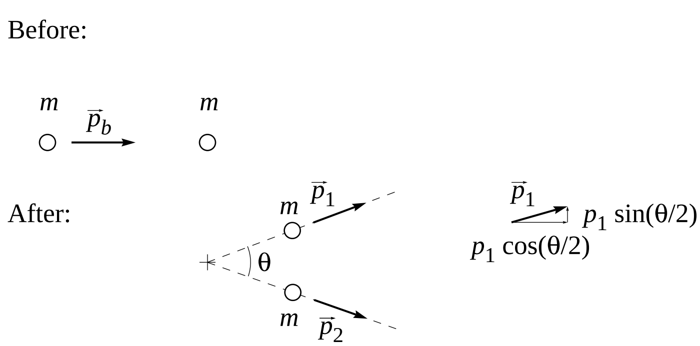
In a classical context, "elastic" means that kinetic energy is conserved. In a relativistic context, energy is always conserved. So what does "elastic" mean in relativity? In a relativistic collision a particle might change its mass, or particles might be created, or destroyed. If these things *don't* happen, then the collision is called elastic.

Experimentally, we can realize such a collision by accelerating a proton to high speed, then aiming it at a stationary hydrogen atom. You might object that the hydrogen atom is not just a proton… it has an electron attached. Or you might object that no hydrogen atom is stationary… because of finite temperature the target atom will be jiggling around. Both objections are legitimate. But if the projectile proton is moving relativistically it has kinetic energy about equal to its rest energy, namely

$$m_{\mathrm{p}}c^2 = 0.938 \text{ GeV} = 0.938 \times 10^9 \text{ eV}. \tag{6.6}$$

In contrast, the electron is bound to the proton by an energy of about 13 eV, and thermal energies at room temperature are about $\frac{1}{40}$ eV. These energies are so small compared to the kinetic energy of the projectile that the associated effects can be safely ignored. So, as a matter of fact, the result of the collision will be two protons and one electron flying apart, but we ignore the electron. (We also ignore the energies due to electrostatic repulsion of the protons. This is because we are, as always, thinking of initial and final states with protons so far apart that the electrostatic interaction is not significant.)

To keep the algebra from growing too complicated, we consider here only the case in which the two protons fly off symmetrically:



According to conservation of momentum

$$\vec{p}_b = \vec{p}_1 + \vec{p}_2 \tag{6.7}$$

while according to conservation of energy

$$E_b + m_{\rm p}c^2 = E_1 + E_2. \tag{6.8}$$

The momentum conservation equation bundles two equations: Conservation of horizontal momentum

$$p_b = p_1 \cos(\theta/2) + p_2 \cos(\theta/2) \tag{6.9}$$

and conservation of vertical momentum

$$0 = p_1 \sin(\theta/2) - p_2 \sin(\theta/2). \tag{6.10}$$

This second equation implies that $p_1 = p_2$ whence $E_1 = E_2$.

As a result of this, we can write conservation of horizontal momentum as

$$p_b = 2p_1 \cos(\theta/2) \tag{6.11}$$

and conservation of energy as

$$E_b + m_{\rm p}c^2 = 2E_1. \tag{6.12}$$

We desire to

find θ in terms of p_b and E_b
by eliminating E_1 and p_1 from these equations
keeping in mind that we don't want to introduce initial or final velocity.

Now we've done the physics, and we have our objectives clearly in mind. It's time to turn on the math. Since we want to solve for θ, let's do that:

$$\cos(\theta/2) = \frac{p_b}{2p_1}. \tag{6.13}$$

In order to avoid introducing velocities we'll use

$$\begin{aligned} E_b^2 - (p_b c)^2 &= (m_{\rm p}c^2)^2 \\ E_1^2 - (p_1 c)^2 &= (m_{\rm p}c^2)^2 \end{aligned}$$

and this suggests that we should square both sides of equation (6.13):

$$\begin{aligned} \cos^2(\theta/2) &= \frac{p_b^2}{4p_1^2} \\ &= \frac{E_b^2 - (m_{\rm p}c^2)^2}{4[E_1^2 - (m_{\rm p}c^2)^2]} \\ &= \frac{E_b^2 - (m_{\rm p}c^2)^2}{(E_b + m_{\rm p}c^2)^2 - 4(m_{\rm p}c^2)^2}. \end{aligned} \tag{6.14}$$

We have achieved our objective of finding θ in terms of initial quantities!

We could stop here, but doing some algebraic cleanup will make our result a lot easier to understand and to work with. First, it gets tedious to write, over and over again, the expressions E_b and $m_\mathrm{p}c^2$. Since we no longer have E_1 around to confuse things, I'll use E and M as shorthand for these quantities. Second, do you remember your half-angle formulas? Neither do I. But I know where to look them up, and one of them says that $\cos^2(\theta/2) = \frac{1}{2}(\cos\theta + 1)$. Thus our last equation becomes

$$\begin{aligned}
\tfrac{1}{2}(\cos\theta + 1) &= \frac{E^2 - M^2}{E^2 + 2EM - 3M^2} \\
\cos\theta &= \frac{2E^2 - 2M^2 - (E^2 + 2EM - 3M^2)}{E^2 + 2EM - 3M^2} \\
&= \frac{E^2 - 2EM + M^2}{E^2 + 2EM - 3M^2} \\
&= \frac{(E - M)^2}{(E - M)(E + 3M)} \\
&= \frac{E - M}{E + 3M}.
\end{aligned}$$

Removing the shorthand, our final result is

$$\cos\theta = \frac{E_b - m_\mathrm{p}c^2}{E_b + 3m_\mathrm{p}c^2}. \tag{6.15}$$

Mathematicians stop at the last equation and say "This is it!" Physicists never do. Instead, we try to see what the last equation is trying to tell us about nature. Consider the classical case where the total energy E_b is just a bit more than the rest energy $m_\mathrm{p}c^2$, i.e. when $E_b = m_\mathrm{p}c^2 + \epsilon$ and $\epsilon \ll m_\mathrm{p}c^2$. In this case

$$\cos\theta \approx \frac{\epsilon}{4m_\mathrm{p}c^2} \approx 0 \tag{6.16}$$

or $\theta \approx 90°$. We have recovered the classical result!

What about the "ultrarelativistic" case $E_b \gg m_\mathrm{p}c^2$? In this case

$$\cos\theta \approx \frac{E_b}{E_b} = 1 \tag{6.17}$$

or $\theta \approx 0°$. When the projectile energy grows very large, the separation angle becomes very small.

6.3 Particle creation

The fact that the sum over particle masses of a system is *not* always conserved results in dramatic consequences. One is that the mass of a glob includes contributions from the thermal, rotational, and oscillational energy of the glob and its components. Another is that new particles can be created. For example, if a projectile proton collides with a target proton with sufficient energy, the outgoing particles might be the two initial protons plus some additional particles created in the collision! This kind of collision is called "inelastic" in a relativistic context.

You might think, for example, that the reaction could be

$$\mathrm{p} + \mathrm{p} \to \mathrm{p} + \mathrm{p} + \mathrm{p}.$$

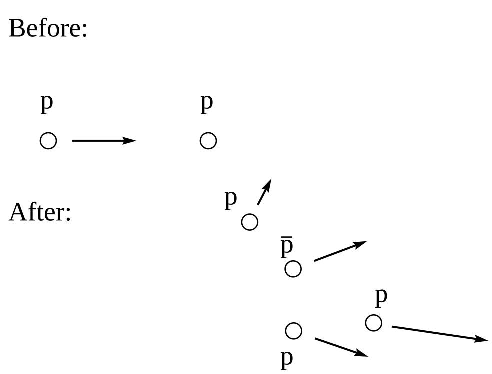
As far as conservation of energy and momentum goes, this is a perfectly feasible reaction (if the projectile proton has energy high enough). But in fact it is never observed: it would violate conservation of charge. Well then, perhaps this reaction

$$\mathrm{p} + \mathrm{p} \to \mathrm{p} + \mathrm{p} + \mathrm{p} + \mathrm{e}.$$

could happen? In fact, this reaction doesn't happen either: it violates a different law called "conservation of lepton number". However the reaction

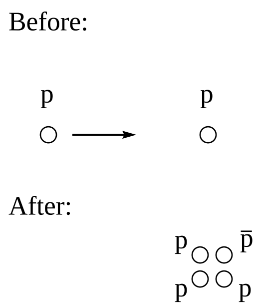
$$\mathrm{p} + \mathrm{p} \to \mathrm{p} + \mathrm{p} + \mathrm{p} + \bar{\mathrm{p}},$$

does occur. In this expression $\bar{\mathrm{p}}$ represents the so-called antiproton: a particle with exactly the same mass as a proton, but with the opposite charge and all other properties. Other reactions might result through this collision (for example, at sufficiently high projectile energies, a collision can result in the formation of a neutron-antineutron pair, or of two proton-antiproton pairs) but the reaction producing one proton and one antiproton is the one we'll investigate.



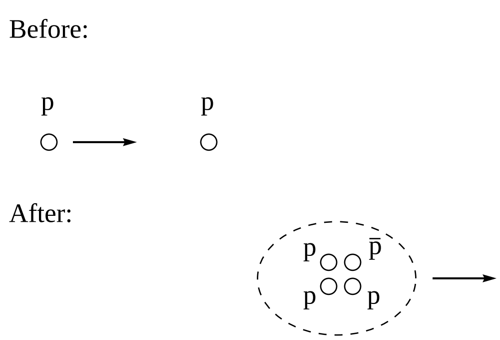
There are a lot of questions we could ask about this reaction: What is the probability of this reaction (as opposed to elastic scattering) happening? If we know the exit angle of three particles, what is the exit angle of the remaining particle? But the question we'll ask is: What is the smallest incoming projectile energy for which this reaction will occur? To find this, the so-called threshold energy, we look for the final situation in which the four particles exit with the minimum possible energies.

Because any particle's minimum possible energy is its rest energy, your first thought might be that threshold would occur when the four product particles are all stationary:



In this scenario the kinetic energy of the incoming projectile is completely converted into the rest energy of the created proton and antiproton. Hence at threshold the incoming projectile would need kinetic energy $2m_{\rm p}c^2$ or total energy $3m_{\rm p}c^2$.

This scenario, however, is not correct. While energy is conserved, momentum is not: the initial situation has some momentum, the final situation has zero momentum. In truth, at threshold the four product particles are not stationary: instead they have the smallest possible velocities consistent with momentum conservation. A moment's consideration will convince you that in this case the four exit particles will have no velocity relative to each other. That is, at threshold the four product particles will move together as a glob.



We could analyze this situation by writing down energy conservation and momentum conservation in the laboratory frame (the one shown here). However the conserved invariant provides a shortcut that is not only mathematically easier but physically more insightful. Here are the energies and momenta tabulated in two different frames:

	Before: lab frame	After: glob's frame
$E^{\rm total}$	$E_b + m_{\rm p}c^2$	$4m_{\rm p}c^2$
$p^{\rm total}$	p_b	0

Now, energy and momentum are conserved over time in any single frame, but not across frames. However the conserved invariant $(E^{\text{total}})^2 - (p^{\text{total}}c)^2$ is constant not only over time but also across frames.

Evaluating the conserved invariant in the lab frame before the collision and in the glob's frame after the collision, we find

$$\begin{aligned}
(E_b + m_{\rm p}c^2)^2 - (p_b c)^2 &= (4m_{\rm p}c^2)^2 \\
E_b^2 + 2E_b m_{\rm p}c^2 + (m_{\rm p}c^2)^2 - (p_b c)^2 &= (4m_{\rm p}c^2)^2 \\
\left[E_b^2 - (p_b c)^2\right] + 2E_b m_{\rm p}c^2 + (m_{\rm p}c^2)^2 &= (4m_{\rm p}c^2)^2 \\
(m_{\rm p}c^2)^2 + 2E_b m_{\rm p}c^2 + (m_{\rm p}c^2)^2 &= (4m_{\rm p}c^2)^2 \\
2E_b m_{\rm p}c^2 + 2(m_{\rm p}c^2)^2 &= 16(m_{\rm p}c^2)^2 \\
E_b &= 7(m_{\rm p}c^2). \qquad (6.18)
\end{aligned}$$

That's the threshold energy. The initial projectile proton must have total energy seven times its rest energy (i.e. kinetic energy six times its rest energy) in order to strike a stationary target proton an produce a proton-antiproton pair. If the projectile proton enters with energy greater than threshold, then the four exiting particles will not be at rest in their glob frame, but instead will fly away from each other.

Problems

6.1. *Angle squeeze.* We analyzed equation (6.15) to show that when E_b is very small $\theta \to 90^\circ$ and that when E_b is very large $\theta \to 0^\circ$. Deduce in addition that as E_b increases, θ decreases *monotonically.*

6.2. *Particle creation, I.* The Fermilab accelerator in Batavia, Illinois, gives a proton a total relativistic energy (i.e. rest energy plus kinetic energy) of 300 GeV. That high-speed proton is then directed toward a stationary proton. The resulting collision can produce a new particle X through the reaction

$$\rm p + p \longrightarrow p + p + X$$

What is the largest possible rest mass $M_{\rm X}$ of a particle created in this way?

6.3. *Particle creation, II.* We have discussed the creation of a proton-antiproton pair by shooting a fast proton at a stationary proton. The same creation can be accomplished by shooting a fast electron at a stationary proton. What is the energy of the lowest energy electron that can perform this feat? (A proton has almost 2000 times the mass of an electron, so the mass of an electron can be neglected relative to the mass of a proton: e.g. use $m_{\rm e} + 3m_{\rm p} \approx 3m_{\rm p}$.)

6.4. *Particle pair creation.* A moving proton can collide with a stationary proton to create a charged pion-antipion pair:

$$\mathrm{p}^+(\text{in motion}) + \mathrm{p}^+ \to \mathrm{p}^+ + \mathrm{p}^+ + \pi^- + \pi^+.$$

What minimum incoming proton energy (in MeV) will allow this process to happen? (For a proton, $m_\mathrm{p}c^2 = 938$ MeV, for a charged pion, $m_\pi c^2 = 139$ MeV.)

6.5. *Speed at threshold.* A proton with the threshold energy given by equation (6.18) has what velocity? (To high precision, the arithmetic can be done in your head.)

6.6. *Can it be?* Show that the following processes are impossible:

a. A free electron absorbs a photon. (Note: An electron always has rest mass m_e . . . there is no "excited state" of an electron with larger mass.)

b. A single photon in empty space transforms into an electron and a positron.

c. A fast positron and a stationary electron annihilate, producing a single photon.

6.7. *Photon absorption.* A stationary, ground state atom of mass m_g absorbs a photon of energy E_γ. What is the mass of the resulting excited atom?

6.8. *Nuclear decay.* A stationary excited nucleus decays to its ground state by emitting a gamma-ray photon of energy E_γ. The ground state nucleus recoils in the opposite direction at speed v. Show that when $v \ll c$ the change of mass of the nucleus is approximately

$$m_e - m_g \approx \frac{E_\gamma}{c^2}\left[1 + \tfrac{1}{2}(v/c)\right].$$

(The exact same phenomena occurs when an excited atom emits a light photon, but in this case the change of mass is usually so small that it's not measurable.) Note that the mass change is *more* than E/c^2 . . . another example showing that the naive idea of "mass is converted into energy through $E = mc^2$" is useful for a general impression but not precisely correct.

6.9. *Decay of a π^0 meson.* A neutral π^0 meson (mass $m_\pi c^2 = 135$ MeV) decays into two photons and nothing else. A π^0 meson of total energy 973 MeV decays and the resulting photons move in opposite directions along the π^0 meson's original line of motion.

a. (8 points) What is the energy of the more energetic photon? (*Clue:* First prove that if the resulting photons have energy E_1 and E_2, then $4E_1E_2 = (m_\pi c^2)^2$.)

b. (2 points) Does the more energetic photon move in the direction that the π^0 meson was heading, or in the opposite direction? (*Clue:* See problem 5.6, *Photon energy.*)

6.10. *Nuclear fission, I.* A nucleus of mass m_N, stationary in the laboratory, fissions into two daughter nuclei of mass m_A and m_B. Daughter A emerges going north. You are responsible for aligning and calibrating a detector for daughter B.

a. (2 points) Where should you position the detector?

b. (8 points) Find an expression for the energy of daughter B in terms of the three masses m_N, m_A, and m_B.

6.11. *Nuclear fission, II.* Here is a physics problem that you are *not* supposed to solve:

> A stationary nucleus of mass m_N fissions into two identical daughter nuclei each of mass $f\frac{1}{2}m_N$. What is the momentum of each daughter, as a function of f and m_N?

Four friends work this problem independently. When they get together afterwards to compare results, they find that they have produced four different answers! Their candidate answers are

(a) $\sqrt{1-f^2}\,\frac{1}{2}m_N c^2$

(b) $\sqrt{f^2-1}\,\frac{1}{2}m_N c$

(c) $\sqrt{1-f^2}\,\frac{1}{2}m_N c$

(d) $\sqrt{1-\frac{1}{2}f^2}\,m_N c$

Provide simple reasons showing that three of these candidate answers must be incorrect. (No one would say their computer program was finished without testing and debugging their first attempt. The same is true for a physics problem: this problem suggests ways to do that testing.)

⟦*Answers:* Candidate (a) does not have the correct dimensions for momentum. For the perfectly reasonable value $f=\frac{1}{2}$, candidate (b) gives an imaginary momentum! There are no problems with candidate (c). When $f=1$, there is no mass loss to "convert" into energy, so the resulting momentum must vanish; candidate (d) fails this test.⟧

6.12. *Pick up an electron.* An electron of mass m_e and energy E_b strikes a stationary atom of mass m_A. The atom absorbs the electron and emerges from the collision as a negatively charged ion. [It can happen that as the atom absorbs the electron, it also emits one or more photons (the energy of the emitted photons is called "electron affinity"). This problem treats the case when such emission does *not* happen.]

a. Show that the mass of that ion is given through

$$m_{\text{ion}}^2 = m_e^2 + 2(E_b/c^2)m_A + m_A^2.$$

b. Interpret this formula by comparing m_{ion} to the classical expectation $m_{ce} = m_e + m_A$, using the kinetic energy of the incoming electron $\text{KE} = E_b - m_e c^2$. Moral of the story: the "conversion of energy to mass" described by the formula $E = mc^2$ holds only in the zero-momentum frame.

6.13. *Cosmic ray cutoff.* The universe is filled with protons traveling in random directions ... these are called cosmic rays. It is also filled with the "3 K background radiation," i.e. photons of temperature 3 K (corresponding to $E_\gamma = 2.5\times10^{-10}$ MeV). A cosmic ray of high energy can interact with such a photon to produce a neutral π-meson through $\gamma + \text{p} \to \text{p} + \pi$. Assume that this collision is head on, and show that the reaction can occur only if the incoming proton has an energy of E_p^X or more, where

$$E_\text{p}^X + \sqrt{(E_\text{p}^X)^2 - M_\text{p}^2} = \frac{M_\pi^2 + 2M_\text{p}M_\pi}{2E_\gamma}.$$

(The symbols M_p and M_π stand for $m_\text{p}c^2 = 938$ MeV and $m_\pi c^2 = 135$ MeV.) Evaluate E_p^X numerically by noting that $E_\text{p}^X \gg M_\text{p}$. (This effect probably accounts for the so-called Greisen-Zatsepin-Kuzmin cut-off in the observed cosmic ray energy spectrum near this energy.)

Chapter 7

Force

We've been talking a lot about energy and momentum, and not so much about force. Let's do that now.

7.1 The effect of a force

There are several ways that the familiar Newtonian laws, such as the second law

$$\vec{F}^{\text{net}} = m\frac{d^2\vec{r}}{dt^2} = \frac{d\vec{p}}{dt}. \tag{7.1}$$

could extend to relativity. The obvious candidates are

$$\vec{F}^{\text{net}} = m\frac{d^2\vec{r}}{dt^2} \quad \text{or} \quad \vec{F}^{\text{net}} = m\frac{d^2\vec{r}}{d\tau^2} \quad \text{or} \quad \vec{F}^{\text{net}} = \frac{d\vec{p}}{dt} \quad \text{or} \quad \vec{F}^{\text{net}} = \frac{d\vec{p}}{d\tau}. \tag{7.2}$$

As always, the question of which one works is a question for experiment to answer. Here's the way that works.

In any given inertial frame, the net force on a particle is related to the momentum through

$$\vec{F}^{\text{net}} = \frac{d\vec{p}}{dt}. \tag{7.3}$$

(Each inertial frame will have different values for $\vec{p}$, for t, and for $\vec{F}^{\text{net}}$, but in every frame they are related through this equality.) For example, if the particle has charge q and is subject to electric field $\vec{E}$ and magnetic field $\vec{B}$, and to no other forces, then

$$q\left[\vec{E} + \vec{v} \times \vec{B}\right] = \frac{d\vec{p}}{dt}. \tag{7.4}$$

(Having reminded you that $\vec{F}^{\text{net}}$ stands for the sum of all forces acting on the particle, the net force, I'm going to drop the annoying superscript "net". Every $\vec{F}$ in this chapter means "net force".)

Our job now is to find how velocity (not momentum) responds to net force. We do so using only the relations

$$\frac{\vec{v}}{c} = \frac{\vec{p}c}{E} \tag{7.5}$$

and

$$E^2 - (\vec{p}c)^2 = \text{conserved}. \tag{7.6}$$

(That is, we don't use any relation that mentions mass m.) This second relation requires a bit of explanation: There *is* an external force, so in any particular frame energy and momentum are *not* conserved. However an increase in E is balanced by the increase in pc, in such a way that the combination $E^2 - (\vec{p}c)^2$ is conserved. Because this combination is invariant, it is equal to its value in the particle's rest frame, namely $(mc^2)^2$, which doesn't change with time. (Thus, this derivation holds for an electron that has the same mass always; it won't hold for a wad of bubble gum with increasing temperature, because that wad has an increasing mass.)

The time derivative of equation (7.5) is

$$\frac{d\vec{v}/c}{dt} = \frac{(d\vec{p}/dt)c}{E} - \frac{\vec{p}c}{E^2}\frac{dE}{dt}.$$

But the time derivative of equation (7.6) is

$$2E\frac{dE}{dt} - 2\vec{p}c \cdot \frac{d\vec{p}c}{dt} = 0$$

so

$$E\frac{dE}{dt} = \vec{p}c \cdot \frac{d\vec{p}c}{dt} = \vec{p}c^2 \cdot \vec{F}.$$

Thus

$$\frac{d\vec{v}/c}{dt} = \frac{\vec{F}c}{E} - \frac{\vec{p}c}{E^2}\frac{\vec{p}c^2 \cdot \vec{F}}{E}$$

and

$$\frac{E}{c^2}\frac{d\vec{v}}{dt} = \vec{F} - \frac{\vec{p}c}{E}\frac{\vec{p}c}{E} \cdot \vec{F}.$$

In other words

$$\begin{aligned} \frac{E}{c^2}\frac{d\vec{v}}{dt} &= \vec{F} - \frac{\vec{v}}{c}\left(\frac{\vec{v}}{c} \cdot \vec{F}\right) \\ &= (1 - (v/c)^2)\vec{F}_{\parallel} + \vec{F}_{\perp}, \end{aligned} \tag{7.7}$$

where $\vec{F}_{\parallel}$ is the component of $\vec{F}$ parallel to $\vec{v}$ and $\vec{F}_{\perp}$ is the component perpendicular.

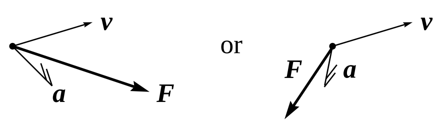
What does this result tell us?

- If the net force is applied parallel to the particle's velocity, then the resulting acceleration is parallel to that force, but the inertia isn't m ... instead it's the "parallel inertia" $(E/c^2)/(1 - (v/c)^2)$.

- If the net force is applied perpendicular to the particle's velocity, then the resulting acceleration is parallel to that force, but the inertia isn't m ... instead it's the "perpendicular inertia" (E/c^2).

- The inertia to parallel forces is *larger* than the inertia to perpendicular forces. In other words, a particle shows less response to a parallel force than to a perpendicular force.

- If the net force is applied neither parallel nor perpendicular to the particle's velocity, then the resulting acceleration is *not* parallel to that force ... the resulting acceleration splays *away* from the particle's velocity axis.



7.2 Starting from rest with a single constant force

Let's apply these results to a concrete situation, one of the first situations you encountered in classical physics, the dropping of a ball from rest. (Here we consider a force constant in the laboratory frame. The case of a force constant in the particle's own rest frame is also interesting, but it's a different problem.)

In the nonrelativistic case the velocity increases linearly with time, $v = (F/m)t$.

What about in relativity? It's pretty straightforward to apply equation (7.7) to this case, because the constant force F is always parallel to the velocity:

$$\frac{E}{c^2}\frac{dv}{dt} = (1-(v/c)^2)F. \tag{7.8}$$

The formula for relativistic energy gives

$$\frac{m}{[1-(v/c)^2]^{1/2}}\frac{dv}{dt} = [1-(v/c)^2]F,$$

whence

$$\frac{m}{[1-(v/c)^2]^{3/2}}\frac{dv}{dt} = F. \tag{7.9}$$

For the nonrelativistic case $v \ll c$, this reduces to the familiar $ma = F$.

How to solve this differential equation? Write it as

$$\frac{1}{[1-(v/c)^2]^{3/2}}dv = \frac{F}{m}dt,$$

then integrate both sides with respect to t finding

$$\int_0^v \frac{1}{[1-(v'/c)^2]^{3/2}} dv' = \frac{F}{m}\int_0^t dt'.$$

The right-hand integral is easy. To prepare for evaluating the left-hand integral I like to use the substitution $v'/c = \beta$, because β is a dimensionless variable. This substitution results in

$$\int_0^{v/c} \frac{1}{[1-\beta^2]^{3/2}} d\beta = \frac{F}{mc}t. \tag{7.10}$$

This is the way I like to see my integrals: all the physical quantities like v and c are worked into the limits of integration, while the integrand and integration variable themselves are pure, naked, and dimensionless.

Now that we've done the setup, we need to execute the integral. There are a number of mathematical tricks you can use, but we're interested in the physics, not the mathematics, so you may merely use a symbolic integrator like Mathematica, or look up the answer in a table of integrals. The result is

$$\left[\frac{\beta}{[1-\beta^2]^{1/2}}\right]_0^{v/c} = \frac{F}{mc}t$$

or

$$\frac{v/c}{\sqrt{1-(v/c)^2}} = \frac{F}{mc}t. \tag{7.11}$$

This answer is correct, but it gives t as a function of v while it's preferable to have v as a function of t. It's straightforward algebra to solve for v finding

$$v = \frac{Ft/m}{\sqrt{1+(Ft/mc)^2}}. \tag{7.12}$$

It's great to get this algebraic answer but we're not finished yet. As scientists we're in the business of studying nature, not equations. What does this equation tell us about nature?

For short values of time, namely $Ft/mc \ll 1$ or

$$t \ll \frac{c}{F/m}, \tag{7.13}$$

this result is to a good approximation $v = (F/m)t$, the classical result. So far, so good. For large values of time, namely $Ft/mc \gg 1$ or

$$t \gg \frac{c}{F/m}, \tag{7.14}$$

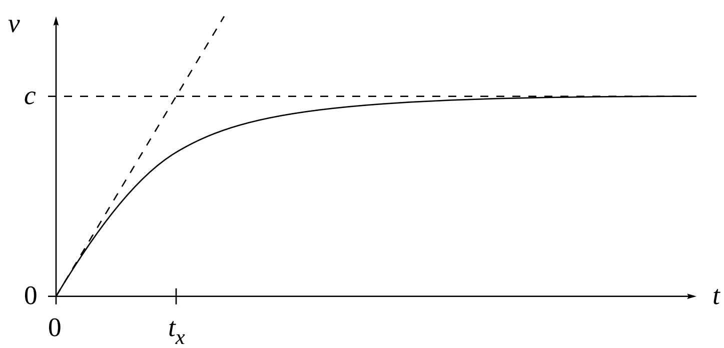
this result is to a good approximation $v = c$ — this is called the "ultra-relativistic limit". If you look at equation (7.12) closely you'll see that v increases monotonically with t, which certainly makes sense physically. Define the "crossover time"

$$t_x = \frac{c}{F/m} \tag{7.15}$$

as an approximate boundary between the classical and the ultra-relativistic limits. With this identification of the characteristic time for the process, the equation can be written in the dimensionally straightforward form

$$\frac{v}{c} = \frac{t/t_x}{\sqrt{1+(t/t_x)^2}}. \tag{7.16}$$

All in all, v as a function of t has the form shown in this graph:



7.3 Why does a high-speed particle exhibit more inertia?

Equation (7.7) shows that a particle of mass m and speed v subject to force F parallel to that velocity obeys

$$F = \frac{m}{[\sqrt{1+(v/c)^2}]^3}\frac{dv}{dt},$$

in contrast to the classical result

$$F = m\frac{dv}{dt}.$$

And you have just shown (equation 7.12) that a particle starting from rest with constant force F has, at time t, the speed

$$v = \frac{(F/m)t}{\sqrt{1+(Ft/mc)^2}}.$$

There was nothing wrong in the derivation of this formula, and it has the satisfying property that the speed v never exceeds c. Yet the derivation seemed to be just mathematics. Can we get a physical handle on why, say, a tennis ball moving near light speed has more inertia than a slow tennis ball?

A very conventional way of getting a tennis ball to move fast is to whack it with a tennis racket. You can make it go faster still by hitting it toward a partner, who whacks it again in the same direction. To get a really fast tennis ball we could implement the following: Build a long shallow trough to hold the tennis ball, hire a whole crew of tennis players to stand by the trough, and tell each athlete to speed up the ball with a whack whenever it goes by.[1] Pretty soon the ball will be going so fast that our athletes won't be able to see it, so we change their instructions to "each second, reach out into the space in front of you and swing your racket." Under these instructions, every second the ball gets one whack (although all but one of our athletes will reach out and swing through empty space). So it seems that every second the ball will add speed, and the speed will increase without limit.

But no. In our reference frame, the athletes are whacking once each second. Not so in the ball's reference frame. In the ball's frame, the athletes's clocks tick slowly, so the whacks come at intervals greater than one second. At first, this interval is just a fraction more than a second. By the time the tennis ball reaches 161 000 miles/second, it feels a whack once every two seconds. So it still picks up speed, but not as readily as it did when it felt a whack once every second.[2] As the ball goes faster and faster, the whacks come slower and slower. It gets to be a minute between whacks, then an hour, then a year, then a century. Naturally, a ball that is being whacked once a century increases its speed at a very low rate. It doesn't pick up speed the way it did back when it was being whacked once a second.

So to the ball, the whacks are coming farther and farther apart. Our athletes are whacking as hard and as often as ever, but they're getting frustrated because they don't see the ball's velocity increasing much. They say the ball is becoming more and more resistant to picking up speed from a whack. The name for "resistance to picking up speed from a whack" is "inertia". Our athletes say that as the ball travels closer and closer to the speed of light, its inertia increases without limit.

These ideas are not just pie in the sky: they have been tested experimentally time and again. You might want to search the Internet for the 1962 film "The Ultimate Speed — An Exploration with High Energy Electrons" by William Bertozzi. (See also William Bertozzi, "Speed and kinetic energy of relativistic electrons," *American Journal of Physics*, **32** (July 1964) 551–555.)

[1] To keep our costs down, it might make sense to build the trough in the form of a circle and hire a limited number of athletes.

[2] In addition to this time dilation, in the ball's frame the athletes are standing closer together and not whacking simultaneously.

7.4 Transformation of a force

In frame F a particle with velocity $\vec{v} \doteq (v_x, v_y, v_z)$ is acted upon by a force $\vec{F} \doteq (F_x, F_y, F_z)$. What are the velocity $\vec{v}'$ and force $\vec{F}'$ in frame F'? The answer comes from

$$\vec{v} = \frac{d\vec{r}}{dt} \quad \text{and} \quad \vec{F} = \frac{d\vec{p}}{dt}. \tag{7.17}$$

We know how to transform $\vec{r}$, t, and $\vec{p}$, so we can figure out how to transform $\vec{v}$ and $\vec{F}$. The calculations are wicked, but the answers are straightforward:

$$v'_x = \frac{v_x - V}{1 - v_x V/c^2} \tag{7.18}$$

$$v'_y = \sqrt{1-(V/c)^2}\,\frac{v_y}{1 - v_x V/c^2} \tag{7.19}$$

$$v'_z = \sqrt{1-(V/c)^2}\,\frac{v_z}{1 - v_x V/c^2} \tag{7.20}$$

$$F'_x = F_x - \frac{V/c}{1 - v_x V/c^2}\left(\frac{v_y}{c}F_y + \frac{v_z}{c}F_z\right) = \frac{F_x - (V\vec{v}/c^2)\cdot\vec{F}}{1 - v_x V/c^2} \tag{7.21}$$

$$F'_y = \sqrt{1-(V/c)^2}\,\frac{F_y}{1 - v_x V/c^2} \tag{7.22}$$

$$F'_z = \sqrt{1-(V/c)^2}\,\frac{F_z}{1 - v_x V/c^2} \tag{7.23}$$

Notice that to find $\vec{F}'$, you must know both $\vec{F}$ and $\vec{v}$. Notice also the special case: If force is applied in only the x-direction, then the force is identical in all frames.

7.5 Four-force

It is a beautiful thing to define a four-force (or Minkowski force)

$$\mathbb{F} = \frac{d\mathbb{p}}{d\tau}, \tag{7.24}$$

and figure out how to use it. In contrast to ordinary vector force, the four-force components transform in a simple, readily-memorable manner (the same as any four-vector), and you don't need to know both velocity and four-force components in one frame to find the four-force components in another frame. However, until one knows how to calculate $\mathbb{F}$ in terms of, say, electric and magnetic fields, this definition is entirely sterile.

Problems

7.1. *$F = ma$.* I have written the relation between acceleration in the form "acceleration = stuff involving force". To cast it in the form "force = stuff involving acceleration" go back to equation (7.3) (for a particle with mass m) and find

$$\vec{F} = \frac{m}{\sqrt{1-(v/c)^2}}\frac{d\vec{v}}{dt} + \frac{m}{\sqrt{(1-(v/c)^2)^3}}\frac{\vec{v}}{c}\left(\frac{\vec{v}}{c}\cdot\frac{d\vec{v}}{dt}\right). \tag{7.25}$$

Write this in a form involving the component of acceleration parallel to $\vec{v}$ and the component of acceleration perpendicular to $\vec{v}$.

7.2. *Qualitative sequence.* Establish the qualitative sequence:

The "parallel inertia" $m/(1-(v/c)^2)^{3/2}$
is greater than the "perpendicular inertia" $m/(1-(v/c)^2)^{1/2}$
which is that same as the "relativistic mass" $m/(1-(v/c)^2)^{1/2}$
which is greater than the "rest mass" m.

In the limit $v \to c$, what happens to the parallel and perpendicular inertias? Does this suggest a mechanism to enforce the law that "no particle can travel at the speed of light or faster"?

7.3. *Crossover time.* Equation (7.12) shows that the crossover from classical behavior ($v \approx (F/m)t$) to ultra-relativisitic behavior ($v \approx c$) occurs near the so-called crossover time $t_x = c/(F/m)$. Evaluate the crossover time numerically for: (a) Acceleration due to gravity near the Earth's surface, $F/m = 9.8$ m/s. (Use the coincidence, accurate to 0.5%, that one year is about $\pi \times 10^7$ seconds.) (b) An electron subject to the modest electric field of 10 V/cm. Do you see the basis for the frequently made claim that "gravity is a weak force"?

7.4. *Force and energy.* Show that for a particle subject to a force,

$$\frac{dE}{dt} = \vec{F}\cdot\vec{v}. \tag{7.26}$$

7.5. *Conservation of energy.* In classical mechanics, conservative forces are important. This is a force on a particle the depends only on the location of the particle (not the time, not the velocity of the particle, etc.) and for which

$$\vec{F}(\vec{r}) = -\vec{\nabla}U(\vec{r}). \tag{7.27}$$

You know that, if a particle is subject to a conservative force, then the total energy

$$\tfrac{1}{2}mv^2 + U(\vec{r}) \tag{7.28}$$

is conserved.

In relativistic mechanics, conservative forces are not so important. In one reference frame, a force might depend only on position. But in another frame, that force will depend on both position *and* time.

But let's consider the situation where, in the lab frame, the massive particle is subject to a conservative force. Show that in this situation the energy

$$\frac{mc^2}{\sqrt{1-(v/c)^2}} + U(\vec{r}) \tag{7.29}$$

is a conserved.

7.6. *Flushing out an error.* The website "Conservapedia" has a page devoted to "Counterexamples to Relativity" (site visited 11 July 2016), which cites the following as a "counterexample to relativity":

> Relativity requires different values for the inertial mass of a moving object: in its direction of motion, and perpendicular to that direction. This contradicts the logical principle that the laws of physics are the same in all directions.

What is Conservapedia's error?

7.7. *Relativistic origin of magnetic force.* Two electrons are stationary in reference frame F. One is located at the origin, and the second is located at $(x = L, y = L, z = 0)$.

a. Show that the electrostatic force on the second electron is $(F_x, F_y, 0)$ where

$$F_x = F_y = \frac{1}{4\pi\epsilon_0}\frac{e^2}{2\sqrt{2}\,L^2}. \tag{7.30}$$

   Thus the force on the second electron points radially away from the first electron.

In reference frame F', the two electrons are moving left, so there are two currents to the right. Thus the forces between them are not only electric, but also magnetic.

b. Use the right-hand rule to show that the magnetic force on the second electron points straight downward. (You can't calculate this force from the Biot-Savart Law, because the Biot-Savart Law applies only to steady currents. But the right-hand rule gives the correct direction.)

c. Use length contraction to show that in frame F', the "radial" direction from the first electron to the second is *closer* to the y-axis than to a diagonal.

d. Given the force and velocity in frame F, transform to find the force in frame F'. Show that the force in frame F' corresponds in direction to a radial electric force plus a downward magnetic force.

From the relativistic point-of-view, the magnetic force is just the transformation of an electrostatic force into a frame in which the source moves.

Chapter 8

Globs

8.1 When does a glob behave like a particle?

Suppose we have a system of particles. (Maybe fields, too.) You remember that in classical mechanics the center of mass of this glob moves exactly like a single particle, subject to the sum of external forces. This is why we can treat a baseball, consisting of billions upon billions of atoms, as a single point particle. Can we find a similar "glob like a particle" result in relativity? In general, the answer is no. But searching for an answer provides us with situations in which we can find such results.

To start off, the center of mass cannot provide this service in relativistic mechanics. Problem 8.1, *Reciprocating cannon balls*, provides an example where there is no external force, yet the center of mass (in some reference frames) accelerates.

A more promising concept is the velocity of the zero-momentum frame. The total momentum of the system (sum over relativistic momentum of each particle and integral over momentum density of each field point) is called $\vec{p}^{\,\text{total}}$ and the total (relativistic) energy is called E^{total}. Using the Lorentz transformation for energy-momentum, you can easily see that the velocity of the zero-momentum frame is

$$\frac{\vec{V}_{ZM}}{c} = \frac{\vec{p}^{\,\text{total}}c}{E^{\text{total}}}. \tag{8.1}$$

Since this is exactly the relation between the velocity of a particle and its momentum and energy, it's a promising candidate for the effective velocity of a glob.

How does this quantity change with time? Consider a system of two particles, 1 and 2, with velocities $\vec{v}_1$ and $\vec{v}_2$, subject to forces $\vec{F}_1$ and $\vec{F}_2$. Then

$$\vec{V}_{ZM} = \frac{\vec{p}_1 + \vec{p}_2}{(E_1 + E_2)/c^2} \tag{8.2}$$

so

$$\begin{aligned}\frac{d\vec{V}_{ZM}}{dt} &= \frac{\vec{F}_1+\vec{F}_2}{(E_1+E_2)/c^2} - \frac{\vec{p}_1+\vec{p}_2}{(E_1+E_2)^2/c^2}\left(\frac{dE_1}{dt}+\frac{dE_2}{dt}\right)\\ &= \frac{\vec{F}_1+\vec{F}_2}{(E_1+E_2)/c^2} - \vec{V}_{ZM}\frac{1}{E_1+E_2}\left(\frac{dE_1}{dt}+\frac{dE_2}{dt}\right).\end{aligned}$$

Now, applying equation (7.26),

$$\frac{E^{\text{total}}}{c^2}\frac{d\vec{V}_{ZM}}{dt} = \vec{F}_1+\vec{F}_2 - \frac{\vec{V}_{ZM}}{c^2}\left(\vec{v}_1\cdot\vec{F}_1+\vec{v}_2\cdot\vec{F}_2\right). \tag{8.3}$$

Comparing this equation to (7.7) shows that $\vec{V}_{ZM}$ does not move *exactly* as a single particle does. In particular, you need to know the velocity of each constituent particle in order to find the acceleration of the zero-momentum frame. However, there are two situations in which the glob acts like a particle:

- In the zero-momentum frame itself, where $\vec{V}_{ZM}=0$.
- If all the constituent velocities are equal. (In which case $\vec{V}_{ZM}=\vec{v}_1=\vec{v}_2$.)

8.2 Finding M^{total} for a glob

In chapter 5, "Momentum, Energy, and Mass", we said that the total mass of a system was

$$M^{\text{total}} = \frac{E^{\text{total}}}{c^2} \text{ in the zero-momentum frame.} \tag{8.4}$$

The total mass M^{total} is the same in all reference frames, but for this equation to work E^{total} must be taken in the zero-momentum frame.

With our deeper knowledge of forces, we reexamine total mass. We will end up confirming our old results, so this section provides insight but no new discovery.

An operational definition of M^{total} for a glob is:

> Move into the zero-momentum frame F and apply a nudge (a small force) to the whole glob or to parts of it. Measured from that same frame F (which is no longer the zero-momentum frame) the glob will now have a small zero-momentum velocity $\delta\vec{V}_{ZM}$ and a small total momentum $\delta\vec{p}^{\text{ total}}$. The total mass of the glob is defined as
>
> $$M^{\text{total}} \equiv \frac{\delta p^{\text{total}}}{\delta V_{ZM}}. \tag{8.5}$$
>
> Because this definition rests upon the classical expression $p = Mv$, it is understood to hold in the limit that δp^{total} and δV_{ZM} both approach zero.

Does this operational definition correspond to the conceptual definition (8.4)? Well, the definition (8.1) of zero-momentum velocity is

$$\frac{\vec{V}_{ZM}}{c} = \frac{\vec{p}^{\,\text{total}} c}{E^{\text{total}}}. \tag{8.6}$$

Applying the quotient rule of differential calculus we find that for any frame

$$\frac{\delta \vec{V}_{ZM}}{c} = \frac{E^{\text{total}}\, \delta \vec{p}^{\,\text{total}} c - \vec{p}^{\,\text{total}} c\, \delta E^{\text{total}}}{(E^{\text{total}})^2}, \tag{8.7}$$

but for the zero-momentum frame

$$\frac{\delta \vec{V}_{ZM}}{c} = \frac{E^{\text{total}}\, \delta \vec{p}^{\,\text{total}} c}{(E^{\text{total}})^2}. \tag{8.8}$$

So the operational definition (8.5)

$$M^{\text{total}} \equiv \frac{\delta p^{\text{total}}}{\delta V_{ZM}} = \frac{E^{\text{total}}}{c^2} \tag{8.9}$$

gives the same result as the conceptual definition (8.4).

This is dramatic. Depending upon the direction and magnitude of the nudge, depending upon whether it's applied to all of the glob or just part if it, the $\delta \vec{V}_{ZM}$ and the $\delta \vec{p}^{\,\text{total}}$ could be very different. But the ratio of the two will be the same for any nudge.

This particular operational definition of mass is called the definition of "inertial mass". There's a different operational definition of mass — the so-called "gravitational mass":

> Put the glob on a balance and measure its gravitational attraction to the earth.

Einstein proposed, and experiment has confirmed, the "principle of equivalence" that these two very different operations always produce the same result: A wad of bubble gum will be harder to accelerate when hot than when cold... and it will be more strongly attracted to the earth when hot than when cold. Similarly for a charged vs. an uncharged capacitor, a fresh vs. a drained battery.

8.3 Through what mechanism does mass increase with energy?

We have seen that the mass of a body must increase with its energy in the zero-momentum frame. But while this demonstration was convincing, it appealed to the intellect and not to the gut. It ignores the question of mechanism. What's really going on? Why does a bottle of gas have more mass when hot than when cold?

Let's look at the simplest model: a gas consisting of two particles, each of mass m, one moving right at speed v_b and the other moving left at speed v_b. By symmetry the net momentum is zero. Nudge both particles to the right by applying equal brief impulses to each particle: each will change velocity by δv, and by symmetry the V_{ZM} is now δv. Each particle changes momentum by just about

$$\delta p = \frac{m\,\delta v}{\sqrt{1-(v_b/c)^2}}. \tag{8.10}$$

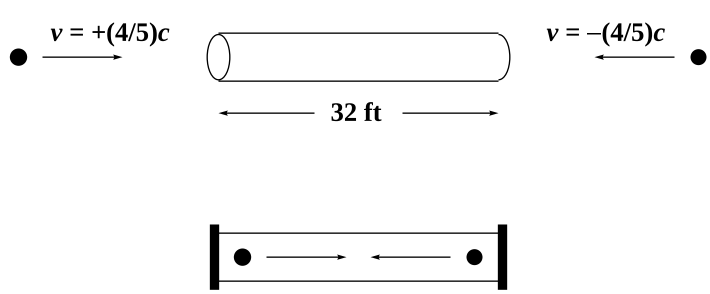
(The momentum change will actually differ somewhat from this, because the speed v changes from v_b during the nudge. But the nudge is small, and the result (8.5) applies to the limit when the nudge becomes very small and this approximation becomes exact.) The total momentum change is twice this. Our operational definition (8.5) says that the total mass is

$$M^{\text{total}} = \frac{2m}{\sqrt{1-(v_b/c)^2}}. \tag{8.11}$$

You see that the total mass increases with the velocity of the two constituent molecules, and that the total mass is E^{total}/c^2.

Problems

8.1. *Reciprocating cannon balls.* [Based on E.F. Taylor and J.A. Wheeler, *Spacetime Physics*, first edition (Freeman, San Francisco, 1963), problem 59.] The following experiment is performed in outer space, far from any stars or planets, so that cannon balls fly in straight lines rather than in gravitationally-inspired parabolas.



Two identical cannon balls are simultaneously launched at speed $v = \frac{4}{5}c$ toward the center of a 32-foot segment of pipe. After they enter the pipe, the two pipe ends are capped. The cannon balls collide elastically at the center of the pipe, bounce back toward the caps, bounce elastically off the caps back toward the center, and so forth, reciprocating without friction.

a. Depict on a space-time diagram the position of each cannon ball as a function of time while reciprocating. (Measure time in terms of the unit "nan", which is the amount of time it takes light to travel one foot. In these units the speed of light is exactly $c = 1$ foot/nan.)

b. This reciprocation is observed from a reference frame moving right at speed $V = \frac{3}{5}c$. Depict the position of each cannon ball as a function of time on a space-time diagram in this frame.

c. Add to both your diagrams the position of the center of mass (midway between the two balls) as a function of time.

Notice that in the pipe's reference frame (part a) the center of mass moves with constant velocity (namely zero). But in the reference frame of part b it regularly changes its velocity even though the system experiences no external force. The "center of mass" velocity doesn't have any simple relation to net external force.

8.2. *Center of mass versus zero momentum frame.* A 2 kg ball travels east at $\frac{4}{5}c$, and a 3 kg ball travels west at $\frac{3}{5}c$. What is the velocity of the center of mass? The velocity of the zero-momentum frame? Notice that these velocities are in opposite directions!

8.3. *Effective glob velocity.* Show that in any frame

$$E^{\text{total}} = \frac{M^{\text{total}}c^2}{\sqrt{1-(V_{ZM}/c)^2}} \quad \text{and} \quad \vec{p}^{\,\text{total}} = \frac{M^{\text{total}}\vec{V}_{ZM}}{\sqrt{1-(V_{ZM}/c)^2}}. \tag{8.12}$$

8.4. *Total mass not conserved.* The total mass of a system is defined through $(E^{\text{total}})^2 - (\vec{p}^{\,\text{total}}c)^2 = (M^{\text{total}}c^2)^2$. Consider a system of two particles, one stationary and the other acted upon by a force. (For example, a neutron and a proton in an electric field.) Show that M^{total} is not constant in time.

8.5. *Center of energy.* The "center of energy" for a particle system is defined as

$$\vec{R}_{\text{coe}} = \frac{\sum_i E_i\vec{r}_i}{\sum_i E_i}. \tag{8.13}$$

For a swarm of particles, each of them free so that the energy of each is constant, show that the velocity of the center of energy is the same as $\vec{V}_{ZM}$.

8.6. *Operational definition of mass.* In some frame, a glob has energy E^{total} and momentum $\vec{p}^{\,\text{total}}$. Some or all of the particles are acted upon by an outside force (which might or might not be small) so that these change to $E^{\text{total}} + \Delta E$ and $\vec{p}^{\,\text{total}} + \Delta\vec{p}$. Using equation (8.1), show that the velocity of the zero-momentum frame changes by

$$\Delta\vec{V}_{ZM} = c^2\frac{\Delta\vec{p} - \vec{p}^{\,\text{total}}(\Delta E/E^{\text{total}})}{E^{\text{total}} + \Delta E}. \tag{8.14}$$

Examine this expression in the case that (a) the frame in question happens to be the zero-momentum frame and (b) the nudge is small. Using operational definition (8.5), what is M^{total}?

8.7. *Electromagnetic energy and momentum.* The electromagnetic field (in vacuum) carries energy density

$$\frac{\epsilon_0}{2}\vec{E}^2(\vec{r},t) + \frac{1}{2\mu_0}\vec{B}^2(\vec{r},t) \tag{8.15}$$

and momentum density

$$\epsilon_0\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t). \tag{8.16}$$

Consider a region of volume ΔV which is small enough that $\vec{E}$ and $\vec{B}$ can be taken as constants over the region.

Show that in that region, the invariant combination $E^2 - (pc)^2$ is

$$\left(\frac{\epsilon_0^2}{4}E^4 + \frac{\epsilon_0^2}{2\mu_0}E^2B^2 + \frac{1}{4\mu_0}B^4 - \frac{\epsilon_0}{\mu_0}E^2B^2\sin^2\theta\right)\Delta V^2 \tag{8.17}$$

where θ is the angle between $\vec{E}$ and $\vec{B}$. You'll note that this is not equal to zero. But if the electromagnetic field is due to a free-space wave moving in a single direction, and uniform perpendicular to that direction, then

$$B = \frac{1}{c}E = \sqrt{\epsilon_0\mu_0}\,E \quad \text{and} \quad \theta = 90^\circ. \tag{8.18}$$

Show that in this case, the invariant combination vanishes.

Yakov P. Terletskii, *Paradoxes in the Theory of Relativity* (Plenum Press, New York, 1968) pages 63–64: "Any real light beam has a nonzero proper mass. Only an infinite-plane light wave, i.e., a beam of strictly collinear photons, has a total proper mass zero. But this case of a light beam is almost never realized in practice, because any real light beam is spatially restricted, i.e., it is not an infinite-plane wave."

Chapter 9

Force Laws

9.1 The problem

Previous chapters have treated the question of "What is the effect of a force?" That is, they have generalized the Newtonian formula

$$\vec{F}^{\text{net}} = m\vec{a}.$$

But now we have to treat the question of where forces come from. That is

What force laws can be consistent with relativity?

The most famous force law is Newton's law of gravity, that, for example, the gravitational force on the Earth due to the Sun, separated by distance r, is

$$G\frac{m_E m_S}{r^2}.$$

This says that the force depends on the mass of the Earth, the mass of the Sun, and the distance of separation. Nothing else. In particular, it doesn't depend on the time. If the Sun were to move, then the force exerted on the Earth would change instantly. Such instant message transmission is forbidden by special relativity. Newton's law of gravity, although highly accurate when all relevant velocities are sufficiently low (and when all gravitational fields are sufficiently weak), must in principle be wrong.

The same argument holds for any force law whatsoever when the force depends only on the separation. The spring force law (Hooke's law) $F = -kx$ is a good approximation in some circumstances, but in principle it's wrong. Coulomb's law of electrostatics

$$\frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r^2}$$

is a good approximation in some circumstances, but in principle it's wrong. In short, any force law that involves so-called "action at a distance" cannot be exactly correct: it is wrong in principle.

One consequence of this fact is that, *any force law determined solely through a potential energy*, so that $\vec{F}(\vec{r}) = -\vec{\nabla}U(\vec{r})$, must be wrong in principle. This might discourage you, because your study of classical mechanics has emphasized exactly such force laws. Don't be discouraged because (a) such force laws can be highly reliable approximations and (b) the ideas developed in your classical mechanics courses can guide the discovery of force laws that *are* correct in principle.

⟦By the way, there's another reason that Newton's law of gravity must be wrong in principle: In Newton's scheme the gravitational field is generated solely by mass. But the mass of an object doesn't equal the sum of the mass of its constituents. So if you regarded the source of gravity to be a planet, you'd get one answer. If you regard the source of gravity to be the sum over all the atoms that constitute that planet, you'd get a (slightly) different answer. In any relativistically correct theory of gravity, the source of gravitational field must be not mass alone, but also energy and momentum. (In Einstein's theory of gravity, the source is the so-called "stress-energy four-tensor".)⟧

9.2 A special case of the problem: Hard sphere forces

We know that any force law which depends on separation alone must be wrong. If you're willing to accept this now, then you may move on to the next section for an outline of a solution to this conundrum. This section simply drives the point home by showing how one particular action-at-a-distance force law gives ludicrous results in relativistic situations.

The "hard sphere force law" is a good approximation to the force between two billiard balls. If each ball has radius R, then the force depends on separation r through

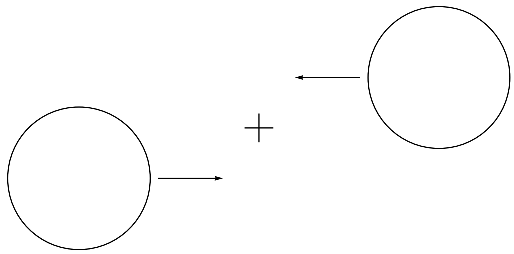
0	when $r > 2R$
infinite repulsion directed along the line between the centers	when $r = 2R$.

The corresponding potential energy is

0	when $r > 2R$
∞	when $r \leq 2R$.

You know how this force law behaves in non-relativistic situations. Suppose two billiard balls approach each other aimed to strike with a 45° blow. (See figure on next page.) They will touch each other and then spring away with a 90° deflection as shown. If the ball on the left entered with a black dot on its upper pole, and the ball on the right entered covered with wet red paint, then, after contact, the ball going down would have a wet red splotch 45° from its black pole dot.

before contact:



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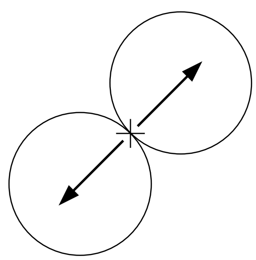
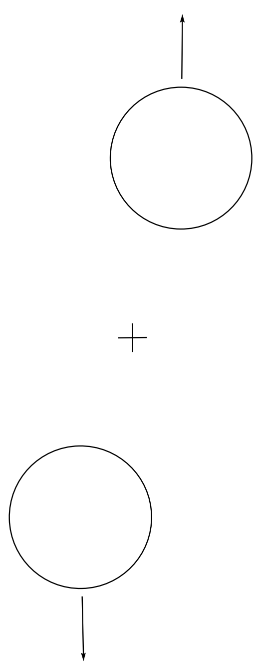
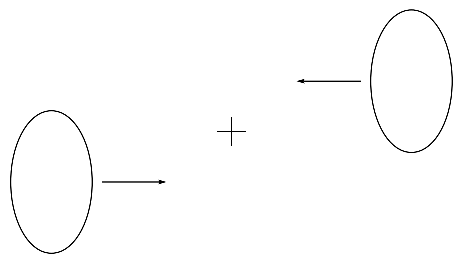


after contact:

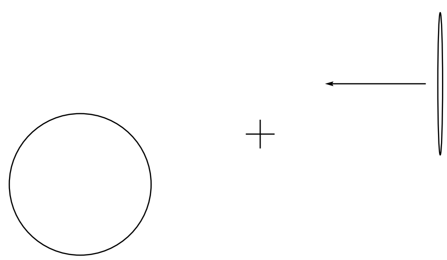


But what if the balls approached each other at relativistic velocities? They would be length contracted in the direction of motion. An analysis in the lab frame suggests a similar symmetry. An analysis in the left ball's frame suggests that the red splotch would be far from the black dot. An analysis in the right ball's frame suggests that the red splotch would be close to the black dot! (Note: the sketch below is not quantitatively accurate.) These three different results cannot all be correct, and this contradiction proves that a hard sphere potential cannot be consistent with special relativity.

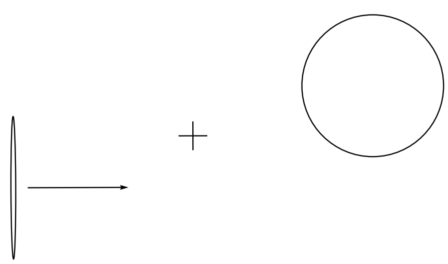
before contact, lab frame:



before contact, left ball's frame:



before contact, right ball's frame:



This should not be a surprise. We have already pointed out that rigid bodies such as these hard spheres are inadmissible in relativity. But it's nice to see exactly how ludicrous the hard sphere force law is.

9.3 The outline of a solution

So, if the familiar action-at-a-distance force laws are not admissible, then what kind of force laws *are* allowed? The general answer to this question is intricate, but I'll give an example to show the character of the solution.

The Coulomb force law of electrostatics allows action-at-a-distance, but you know from your study of electricity and magnetism that the Coulomb force law is not the final word on electromagnetic forces. There is also a velocity-dependent magnetic force, and there are multiple ways to generate electric and magnetic fields: Electric field is generated not only by charge, but also by changing magnetic field. Magnetic field is generated not only by current, but also by changing electric field. These multiple sources of field conspire, through the four Maxwell equations, to ensure that the electric and magnetic fields at one point don't change instantly as the source charges and currents change: instead the effect of source changes propagates outward at the speed of light.

This illustrates the character of the force laws admissible in relativity: The force laws are field theories, there are velocity-dependent forces, and the fields propagate at or below the speed of light.

9.4 Resumé

In the Newtonian formulation of classical mechanics, the force law could be anything: friction, viscous drag, the push of wind that varies capriciously with time, conservative forces, non-conservative forces.

In the Lagrangian or Hamiltonian formulations of classical mechanics, the force law is restricted to conservative forces only. (They can handle velocity-dependent forces of the magnetic kind, because such forces — unlike most velocity-dependent forces — are conservative.) But the conservative force can have any form: a $1/r$ potential, a $1/r^3$ potential, a $\sin(r/r_0)$ potential.

In relativistic mechanics, the force law is further restricted. Only field theories where effects propagate at the speed of light or below are admissible.

It turns out that quantum mechanics restricts the admissible force laws yet again: Most relativistically correct field theories that you could write down are incompatible with quantum mechanics.

So if we demand that our fundamental forces be compatible with both relativity and quantum mechanics, there is only a small cast of possibilities. This is how the law of interaction between quarks — the so-called quantum chromodynamics or QCD — was uncovered. There are so few possible quantum field theories that the law of interaction was found despite minimal experimental input.

Appendix A

Catalog of Misconceptions

The three most common misconceptions concerning special relativity, for students at this level, seem to be:

- The space-time effects of relativity – such as length contraction and time dilation — aren't real, but apparent due to finite speed of light.
- The mass of an object is the sum of the masses of its constituents.
- The formula
$$E = \frac{mc^2}{\sqrt{1-(v/c)^2}}$$
includes potential as well as kinetic energy. (See discussion on page 34.)
- A particle at rest has zero energy. (No one would hold this misconception outright. But it's been my experience that in doing collision problems many people simply forget that in relativity, even a stationary particle carries energy.)