Welcome to Physics 103, "Elementary Physics I". This document presents information about significant figures and about dimensions (or units) that I expect you to understand and to use on a daily basis throughout this course.

— Dan Styer; 1 November 2023

Significant Figures

The impossibility of certainty

What is the pattern of thought most characteristic of science? Is it knowing that "net force causes acceleration rather than velocity"? Is it knowing that"momentum is conserved in the absence of external forces"? Is it knowing that "an object starting at rest with constant acceleration has position change proportional to the time squared"? No, it is none of these three facts — important though they are. The pattern of thought most characteristic of science is knowing that "every observation is imperfect and thus every observed number comes with an uncertainty".

This has an important philosophical consequence: Because all scientific conclusions are based on observations, and all observations contain some uncertainty, no scientific conclusion can be absolutely certain. Anyone worshiping at the alter of science is fooling himself.

This course is more interested in the day-to-day practicalities of uncertainty than in the grand philosophical consequences. How do we express our lack of certainty? How do we work with (add, subtract, multiply) quantities that aren't certain?

Expressing uncertainty

As with any facet of science, the proper approach to uncertainty is not "plug into a formula for error propagation" but instead "think about the issues involved". For example, in the course of building a tree house I measured a plank with a meter stick and found it to be 187.6 cm long. But a more accurate measurement would of course provide a more accurate length: perhaps 187.64722031 cm. I don't know the plank's exact length, I only know an approximate value. In a math class, 187.6 cm means the same as 187.60000000 cm. But in a physics class, 187.6 cm means the same as 187.6??????? cm, where the question marks represent not zeros, but digits that you don't know. The digits that you *do* know are called "significant digits" or "significant figures."

The convention for expressing uncertain quantities is simple: Any digit written down is a significant digit. A plank measured to the nearest millimeter has a length expressed as, say, 103.7 cm or 91.5 cm or 135.0 cm. Note particularly the trailing zero in 135.0 cm: this final digit is significant. The quantity "135.0 cm" is different from "135 cm". The former means "135.0??????? cm", the latter means "135.???????? cm". In the former, the digit in the tenths place is 0, while in the latter, the digit in the tenths place is unknown.

This convention gives rise to a problem for representing large numbers. Suppose the distance between two stakes is 45.6 meters. What is this distance expressed in centimeters? The answer 4560 cm is unsatisfactory, because the trailing zero is not significant and so, according to our rule, should not be written down. This quandary is resolved using exponential notation: 45.6 meters is the same as 4.56×10^3 cm. (This is, unfortunately, one of the world's most widely violated rules.)

Working with uncertainty

Addition and subtraction. I measure a plank with a meter stick and find it to be 187.6 cm long. Then I measure a dowel with a micrometer and find it to be 2.3405 cm in diameter. If I place the dowel next to the plank, how long is the dowel plus plank assembly? You might be tempted to say

```
 187.6
+  2.3405
 --------
 189.9405
```

But no! This is treating the unknown digits in 187.6 cm as if they were zeros, when in fact they're question marks. The proper way to perform the sum is

```
 187.6?????
+  2.3405??
 ----------
 189.9?????
```

So the correct answer, with only significant figures written down, is 189.9 cm.

Multiplication and division. The same question mark technique works for multiplication and division, too. For example, a board measuring 124.3 cm by 5.2 cm has an area given through

```
    1243?
   x  52?
--------
    ?????
   2486?
  6215?
--------
  64????
```

Adjusting the decimal point gives an answer of 6.4×10^2 cm^2

But while the question mark technique works, it's very tedious. (It's even more tedious for division.) Fortunately, the following rule of thumb works as well as the question mark technique and is a lot easier to apply:

> When multiplying or dividing two numbers, round the answer down to the number of significant digits in the least certain of the two numbers.

For example, when multiplying a number with four significant digits by a number with two significant digits, the result should be rounded to two significant digits (as in the example above).

Doing the rounding

Most of the rules for rounding are straightforward. If your result has two significant figures and your calculator gives you the result 1.2317431, you round down to 1.2. If your calculator gives 1.2617431, you round up to 1.3. But what if your calculator gives 1.2517431, given that 1.25 is exactly halfway between 1.20 and 1.30?

Many teachers use the "round to even rule": If the digit to be rounded off is a 5, and the digit that's retained is even, round down. If the digit that's retained is odd, round up. Under this rule 1.2517431 is rounded down to 1.2 (with last digit even), whereas 1.3517431 is rounded up to 1.4 (with last digit even). Because 0 is an even number, 1.0517431 is rounded down to 1.0. Under this rule you round up half the time and you round down half the time, with pleasing symmetry.

I don't insist on the "round to even rule". It gives the misimpression that for any scientific problem there is a single correct answer and a single correct way to find it, that science doesn't involve creativity or ingenuity. In this course, you may round 1.2517431 to either 1.2 or 1.3, I don't care which. But do *not* round it to 1.25, giving you an illegitimate extra digit.

Examples of rounding

Suppose you're working a problem to find a length. You've already determined that the result has three significant digits and the units of meters. This table shows what comes out of your calculator, and then the correct and incorrect results:

Calculator:	1.7432987	1.7032987	1743.2987	1703.2987
Correct:	1.74 m	1.70 m	1.74×10^3 m	1.70×10^3 m
Correct:			1.74 km	1.70 km
Wrong:		1.7 m	1740 m	1700 m

The first wrong answer errs in not reporting a trailing zero that is significant. The second two wrong answers err in reporting a digit (the rightmost zero) that is not significant. In 1700 m, the zero to the left is significant so must be reported, the zero to the right is not significant so must *not* be reported.

Numbers that are certain

Any measured number is uncertain, but *counted* and *defined* quantities can be certain. If there are seven people in a room, there are 7.0000000... people. There are never 7.00395 people in a room. And the inch is *defined* to be exactly 2.5400000... centimeters — there's no uncertainty in this conversion factor, either.

Dimensions

What does “dimensions” mean?

Suppose I say that a table is six feet long or, in symbols,

$$\ell_T = 6 \text{ ft},$$

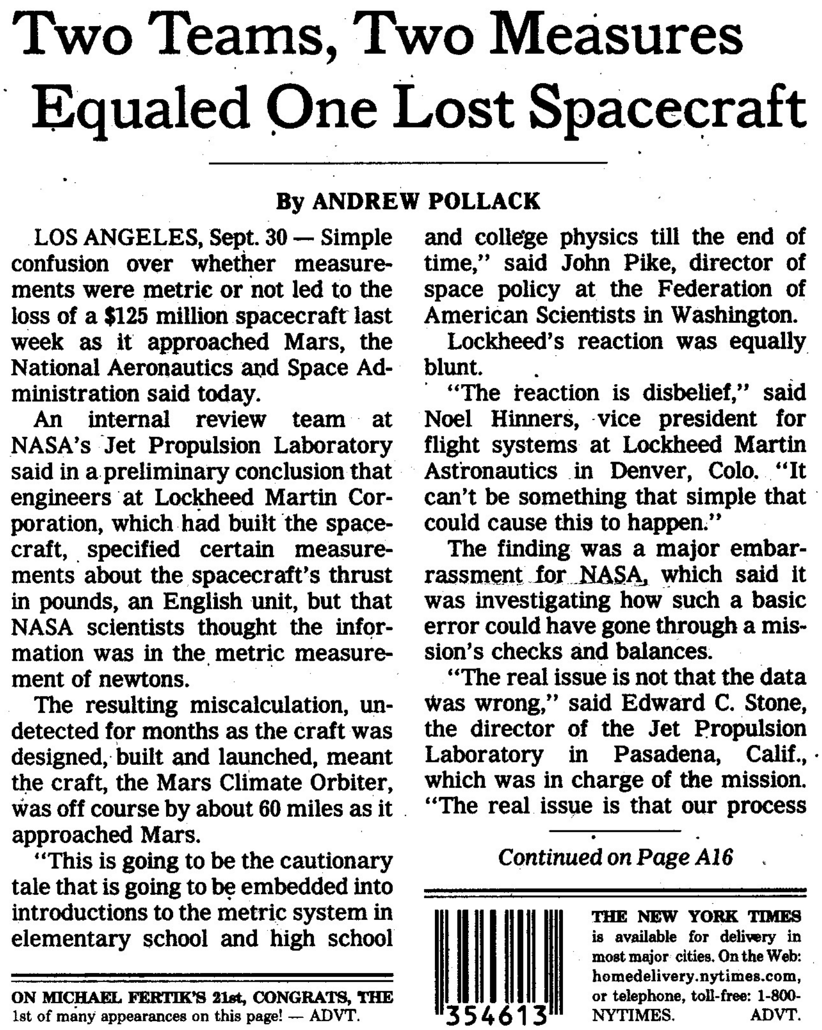
where ℓ_T represents the length of the table. This means that the table is six times as long as the length of the standard foot:

$$\ell_T = 6 \text{ ft} \quad \text{means} \quad \ell_T = 6 \times (\text{the length of the standard foot}).$$

In other words, the symbol “ft” used in the equations above stands for “the length of the standard foot”.

The symbol “ℓ_T” stands for “6 ft”. That is, it stands for the number “six” times the length of the standard foot, or in other words, for the number “six” times the unit “ft”. If I wrote “$\ell_T = 6$” instead of “$\ell_T = 6$ ft”, I’d be dead wrong...just as wrong as if the solution to an algebra problem were “$y = 6x$” and I wrote “$y = 6$”, or if the solution to an arithmetic problem were “6×7” and I wrote “6”. In all three cases, my answer would be wrong because it omitted a factor. (These are, respectively, the factor of “the length of the standard foot”, the factor of x, and the factor of 7.) The length of the table is not 6 — rather, the ratio of the length of the table to the length of the standard foot is 6.

Ignoring the units of a measurement results in practical as well as conceptual error. On 23 September 1999 the “Mars Climate Orbiter” spaceprobe crashed into the surface of Mars, dashing the hopes and dreams of dozens of scientists and resulting in the waste of $125 million. This spacecraft had survived perfectly the long and perilous trip from Earth to Mars. How could it have failed so spectacularly the final, relatively easy, phase of its journey? The manufacturer, Lockheed Martin Corporation, had told the the spacecraft controllers, at NASA’s Jet Propulsion Laboratory, the thrust that the probe’s rockets could produce. But the Lockheed engineers gave the thruster performance data in pounds (the English unit of force), *and they didn’t specify which units they used.* The NASA controllers assumed that the data were in newtons (the metric unit of force).



Two Teams, Two Measures Equaled One Lost Spacecraft

By ANDREW POLLACK

LOS ANGELES, Sept. 30 — Simple confusion over whether measurements were metric or not led to the loss of a $125 million spacecraft last week as it approached Mars, the National Aeronautics and Space Administration said today.

An internal review team at NASA's Jet Propulsion Laboratory said in a preliminary conclusion that engineers at Lockheed Martin Corporation, which had built the spacecraft, specified certain measurements about the spacecraft's thrust in pounds, an English unit, but that NASA scientists thought the information was in the metric measurement of newtons.

The resulting miscalculation, undetected for months as the craft was designed, built and launched, meant the craft, the Mars Climate Orbiter, was off course by about 60 miles as it approached Mars.

"This is going to be the cautionary tale that is going to be embedded into introductions to the metric system in elementary school and high school and college physics till the end of time," said John Pike, director of space policy at the Federation of American Scientists in Washington.

Lockheed's reaction was equally blunt.

"The reaction is disbelief," said Noel Hinners, vice president for flight systems at Lockheed Martin Astronautics in Denver, Colo. "It can't be something that simple that could cause this to happen."

The finding was a major embarrassment for NASA, which said it was investigating how such a basic error could have gone through a mission's checks and balances.

"The real issue is not that the data was wrong," said Edward C. Stone, the director of the Jet Propulsion Laboratory in Pasadena, Calif., which was in charge of the mission. "The real issue is that our process

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New York *Times*, 1 October 1999, page 1, an embarrassing place to have your blunders reported.

Modern information technology actually encourages mistakes like this. When you use a calculator, a spreadsheet, or a computer program, you enter pure numbers like "1.79", rather than quantities like "1.79 feet". So it's especially important to be on your guard and document your units when using computers. Keep the units in your mind, even if you can't keep them in your calculator!

A nitpicky distinction is that the length of the table has the *units* of "feet" and the *dimensions* of "length". If the length of the table were measured in yards or meters it would still have the dimensions of length. But in everyday language the terms "units" and "dimensions" are often used interchangeably.

Unit conversions

It is standard usage to refer to the length of the standard foot by the symbol "ft". But in this document I'll also refer to the length of the standard foot by the symbol ℓ_F. Similarly I'll call the length of the standard yard either "yd" or ℓ_Y.

You know that if a table is 6 feet long it is also 2 yards long:

$$\ell_T = 6 \text{ ft} = 6\ell_F = 2 \text{ yd} = 2\ell_Y.$$

How can this be? It's certainly *not* true that $6 = 2$! It's true instead that 6 ft = 2 yd because the length of a yardstick is three times the length of a foot ruler:

$$\ell_Y = 3\ell_F.$$

This tells us that

$$2 \text{ yd} = 2\ell_Y = 2 \times (3\ell_F) = 6\ell_F = 6 \text{ ft},$$

or, in the opposite direction,

$$6 \text{ ft} = 6\ell_F = 6\ell_F(1) = 6\ell_F\left(\frac{\ell_Y}{3\ell_F}\right) = 6\cancel{\ell_F}\left(\frac{\ell_Y}{3\cancel{\ell_F}}\right) = 2\ell_Y = 2 \text{ yd}.$$

In short, the symbol "ft" can be manipulated exactly like the symbol "ℓ_F", because that's exactly what it means!

Incompatible dimensions

If I walk for 4 yd, and then for 2 ft, how far did I go? The answer is 14 ft or $4\frac{2}{3}$ yd, but not $4 + 2 = 6$ of anything!

If I walk for 4 yd, and then pause for 30 seconds, how far did I go? Certainly *not* 4 yd + 30 sec. The number 34 has no significance in this problem. For example, it can't be converted into minutes.

In general, *you can't add quantities with different units.*

This rule can be quite helpful. Suppose you're working a problem that involves a speed v and a time t, and you're asked to find a distance d. Someone approaches you and whispers: "Here's a hint: use the equation $d = vt + \frac{1}{2}vt^2$." You know that this hint is wrong: The quantity vt has the dimensions of [distance], but the quantity $\frac{1}{2}vt^2$ has the dimensions of [distance×time]. You can't add a quantity with the dimensions of [distance×time] to a quantity with the dimensions of [distance], any more than you could add 30 sec to 4 yd.

Dimensions aren't a cure-all

It's not meaningful to add quantities with different units, but just because two quantities do have the same units doesn't mean it must be meaningful to add them. For example, two quantities from mechanics are work and torque. Both have the dimensions of [force×distance], but the entities are quite distinct and it never makes sense to add a quantity of work to a quantity of torque. Why? Work is defined as a force times a distance parallel to that force, while torque is defined as a force times a distance perpendicular to that force. They are different types of entity, even though they share the same dimensions.

Another example: The rate at which a conveyor belt delivers gravel is measured in kilograms/second. And if a frictional force is proportional to velocity, $F_{\text{friction}} = -bv$, then the friction coefficient b has the units of kilograms/second. But these are clearly entities of different character!

A famous use of dimensional analysis

The following story about the use of dimensional analysis comes from David L. Goodstein, *States of Matter* (Prentice-Hall, 1975, page 436). (See also *Physics Today*, May 2000, page 35; and G.I. Barenblatt, *Scaling Phenomena in Fluid Mechanics* (Cambridge University Press, 1994).)

"Dimensional analysis is a technique by means of which it is possible to learn a great deal about very complicated situations if you can put your finger on the essential features of the problem. An example is the well-known story of how G.I. Taylor was able to deduce the yield of the first nuclear explosion from a series of photographs of the expanding fireball in *Life* magazine. He realized that he was seeing a strong shock expanding into an undisturbed medium. The pictures gave him the radius as a function of time, $r(t)$. All that could be important in determining $r(t)$ was the initial energy release, E, and the density of the undisturbed medium, ρ. The radius, with the dimension of length, depended on E, ρ, and t, and he constructed a distance out of these quantities. E and ρ had to come in as E/ρ to cancel the mass. E/ρ has the dimensions [length]^5/[time]^2, so the only possible combination was

$$r(t) \propto \left(\frac{E}{\rho} t^2\right)^{1/5}.$$

A log-log plot of r versus t (measured from the pictures) gave a slope of $\frac{2}{5}$, which checked the theory, and E/ρ could be obtained from extrapolation to the value of $\log r$ when $\log t = 0$. Since ρ, the density of undisturbed air, was known, E was determined to within a [dimensionless] factor of order one. For the practitioner of the art of dimensional analysis, the nation's deepest secret had been published in *Life* magazine."

Problem

Sound speed. The speed of sound v_s in air, in a given room, could reasonably depend on three things: the air density ρ, the air pressure p, and the room volume V. In other words $v_s = C\rho^x p^y V^z$, where C is some dimensionless number. Use dimensional analysis to determine the exponents x, y, and z.