Atomic units

	conventional basic dimensions (mass, length, time)	unconventional basic dimensions (mass, length, energy = $\mathrm{ML^2/T^2}$)
$\hbar$	$[\mathrm{ML^2/T}]$	$[\mathrm{M^{1/2}LE^{1/2}}]$
m	$[\mathrm{M}]$	$[\mathrm{M}]$
$e^2/4\pi\epsilon_0$	$[\mathrm{ML^3/T^2}]$	$[\mathrm{LE}]$

a. Characteristic energy:
Using unconventional basic dimensions, first combine quantities $\hbar$ and $e^2/4\pi\epsilon_0$ to get rid of [L]. (This is the *only* way to cancel out the [L]s.) This division results in

$$\text{quantity } \frac{\hbar}{e^2/4\pi\epsilon_0} \text{ with dimensions } [\mathrm{M^{1/2}/E^{1/2}}].$$

Square both sides to get

$$\text{quantity } \frac{\hbar^2}{(e^2/4\pi\epsilon_0)^2} \text{ with dimensions } [\mathrm{M/E}].$$

Invert and multiply by m (the *only* way to get rid of the [M]s) to find the *only* characteristic energy,

$$\text{quantity } \frac{m(e^2/4\pi\epsilon_0)^2}{\hbar^2} \text{ with dimensions } [\mathrm{E}].$$

This energy is equal to two Rydberg units (2 Ry).

b. Characteristic time:
Using conventional basic dimensions, first combine quantities $\hbar$ and $e^2/4\pi\epsilon_0$ to get rid of [L]:

$$\text{quantity } \frac{\hbar^3}{(e^2/4\pi\epsilon_0)^2} \text{ with dimensions } \frac{[\mathrm{M^3L^6/T^3}]}{[\mathrm{M^2L^6/T^4}]} = [\mathrm{MT}].$$

Divide by m to get the only quantity with the dimensions of time:

$$\frac{\hbar^3}{m(e^2/4\pi\epsilon_0)^2} \equiv \tau_0.$$

I remember this as

$$\tau_0 = \frac{\hbar}{2\mathrm{Ry}} = 2.4 \times 10^{-17}\ \mathrm{s} = 0.024\ \mathrm{fs}.$$

c. Bonus — Bohr model:
For classical circular orbits,

$$\frac{mv^2}{r} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2}.$$

To this Bohr adds the quantization condition for angular momentum, namely that for the nth Bohr orbit,

$$n\hbar = mv_n r_n.$$

Thus the radius of the nth Bohr orbit comes through

$$\frac{m}{r_n}v_n^2 = \frac{m}{r_n}\left(\frac{n\hbar}{mr_n}\right)^2 = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r_n^2}$$

or

$$\frac{n^2\hbar^2}{mr_n} = e^2/4\pi\epsilon_0$$

whence

$$r_n = n^2\frac{\hbar^2}{m(e^2/4\pi\epsilon_0)} \equiv n^2 a_0.$$

Then the period of the nth Bohr orbit is

$$\begin{aligned}
\text{period}_n &= \frac{\text{circumference}_n}{v_n} \\
&= \frac{2\pi r_n}{n\hbar/mr_n} \\
&= \frac{2\pi m r_n^2}{n\hbar} \\
&= \frac{2\pi m n^4 a_0^2}{n\hbar} \\
&= n^3\frac{2\pi m a_0^2}{\hbar} \\
&= 2\pi n^3\frac{m}{\hbar}\frac{\hbar^4}{m^2(e^2/4\pi\epsilon_0)^2} \\
&= 2\pi n^3\frac{\hbar^3}{m(e^2/4\pi\epsilon_0)^2} \\
&= 2\pi n^3\tau_0
\end{aligned}$$

d. Heartbeats to orbits:
The average person lives about 80 years. The average heartbeat lasts about one second. The number of seconds in a year is surprisingly close to $\pi \times 10^7$. Thus the average heart beats about 3×10^9 times. (This three billion beats represents spectacular performance: the fuel pump in a car can't do nearly as well.)

How long does it take an innermost electron to execute one Bohr orbit?
The orbital time is $2\pi\tau_0$ or about 1.5×10^{-16} s.

How long does it take this electron to execute three billion orbits (a "lifetime's worth")?
About 5×10^{-7} s.

So how many "atom lifetimes" pass in one second?
About $1/(5 \times 10^{-7})$ or two million.

e. The time-dependent Schrödinger equation in scaled variables:
For any function $f(x)$ we have

$$\begin{aligned}
\frac{\partial f(x)}{\partial x} &= \frac{\partial f(\tilde{x})}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x} = \frac{\partial f(\tilde{x})}{\partial \tilde{x}}\frac{1}{a_0} \\
\frac{\partial^2 f(x)}{\partial x^2} &= \frac{\partial}{\partial \tilde{x}}\left[\frac{\partial f(\tilde{x})}{\partial \tilde{x}}\frac{1}{a_0}\right]\frac{\partial \tilde{x}}{\partial x} = \frac{\partial^2 f(\tilde{x})}{\partial \tilde{x}^2}\frac{1}{a_0^2}.
\end{aligned}$$

Apply this to the time-dependent Schrödinger equation:

$$
\begin{aligned}
i\hbar\frac{\partial\Psi}{\partial t} &= -\frac{\hbar^2}{2m}\nabla^2\Psi - \frac{e^2}{4\pi\epsilon_0}\frac{1}{r}\Psi \\
i\hbar\frac{1}{\tau_0}\frac{\partial\Psi}{\partial\tilde{t}} &= -\frac{\hbar^2}{2m}\frac{1}{a_0^2}\widetilde{\nabla}^2\Psi - \frac{e^2}{4\pi\epsilon_0}\frac{1}{a_0\tilde{r}}\Psi \\
i\frac{\partial\Psi}{\partial\tilde{t}} &= -\frac{1}{2}\left[\frac{\hbar}{m}\frac{\tau_0}{a_0^2}\right]\widetilde{\nabla}^2\Psi - \left[\frac{\tau_0}{\hbar}\frac{e^2}{4\pi\epsilon_0}\frac{1}{a_0}\right]\frac{1}{\tilde{r}}\Psi.
\end{aligned}
$$

However, quick perusal of the definitions of a_0 and τ_0 will convince you that both of the expressions in square brackets are equal to 1! Thus

$$i\frac{\partial\Psi}{\partial\tilde{t}} = -\frac{1}{2}\widetilde{\nabla}^2\Psi - \frac{1}{\tilde{r}}\Psi.$$

Multiply both sides by $a_0^{3/2}$, because $\widetilde{\Psi} = a_0^{3/2}\Psi$, to get

$$i\frac{\partial\widetilde{\Psi}}{\partial\tilde{t}} = -\frac{1}{2}\widetilde{\nabla}^2\widetilde{\Psi} - \frac{1}{\tilde{r}}\widetilde{\Psi}.$$

Grading:

2 points for part **a**
2 points for part **b**
2 points extra for part **c**
3 points for part **d**
3 points for part **e**