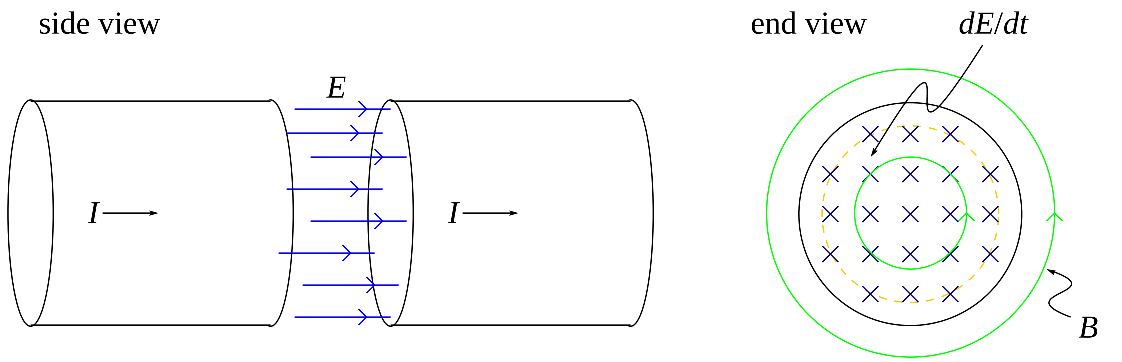
Capacitor fields and energy

Griffiths, *Electrodynamics*, fourth edition, problem 7.34: Charging a capacitor



The changing $\vec{E}$ makes $\vec{B}$... in fact, $\epsilon_0 \dfrac{\partial \vec{E}}{\partial t}$ makes $\vec{B}$ in exactly the same way that $\vec{J}$ makes $\vec{B}$ (see end view sketch), because

$$\begin{aligned}
\vec{\nabla} \times \vec{B} &= \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \\
\oint \vec{B} \cdot d\vec{\ell} &= \mu_0 I_{\text{embraced}} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}
\end{aligned}$$

Apply the integral form of the Ampere-Maxwell law to the dashed yellow circle of radius s.

$$\begin{aligned}
\oint \vec{B} \cdot d\vec{\ell} &= \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \\
B(2\pi s) &= \mu_0 \epsilon_0 \frac{d}{dt}(E\pi s^2) \\
B &= \frac{\mu_0 \epsilon_0}{2} s \frac{dE}{dt}
\end{aligned}$$

But within the capacitor

$$E = \frac{\sigma}{\epsilon_0} = \frac{q/A}{\epsilon_0} = \frac{q}{\epsilon_0 \pi a^2}$$

so

$$\frac{dE}{dt} = \frac{1}{\epsilon_0 \pi a^2} \frac{dq}{dt} = \frac{1}{\epsilon_0 \pi a^2} I$$

and

$$B = \frac{\mu_0}{2\pi a^2} s I.$$

Griffiths, *Electrodynamics*, fourth edition, problem 8.2: Energy in charging a capacitor

(a)

$$E = \frac{It}{\epsilon_0 \pi a^2} \qquad B = \frac{\mu_0}{2\pi a^2} s I$$

(b)

$$u_{em} = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) = \frac{I^2}{2\pi^2 a^4}\left(\frac{t^2}{\epsilon_0} + \frac{\mu_0 s^2}{4}\right)$$

The Poynting vector

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

points everywhere inward, that is

$$\vec{S} = -S\hat{s},$$

and has magnitude

$$S = \frac{1}{\mu_0}EB = \frac{I^2 st}{\epsilon_0 2\pi^2 a^4}.$$

Check: (Use expression for divergence in cylindrical coordinates.)

$$\begin{aligned}
\frac{\partial u_{em}}{\partial t} &= -\vec{\nabla}\cdot\vec{S} \\
\frac{\partial u_{em}}{\partial t} &= \frac{1}{s}\frac{\partial}{\partial s}(sS) \\
\frac{I^2}{2\pi^2 a^4}\left(\frac{2t}{\epsilon_0}\right) &= \frac{1}{s}\frac{\partial}{\partial s}\left(\frac{I^2 s^2 t}{\epsilon_0 2\pi^2 a^4}\right) = \frac{1}{s}\left(\frac{I^2 2st}{\epsilon_0 2\pi^2 a^4}\right) = \frac{I^2 t}{\epsilon_0 \pi^2 a^4}
\end{aligned}$$

(c) Electric energy is

$$\frac{\epsilon_0}{2}\int E^2\, d^3r = \frac{\epsilon_0}{2}\left(\frac{It}{\epsilon_0 \pi a^2}\right)^2 (w\pi a^2) = \frac{I^2 t^2 w}{\epsilon_0 2\pi a^2}.$$

Magnetic energy is

$$\frac{1}{2\mu_0}\int B^2\, d^3r = \frac{1}{2\mu_0}\int_0^a ds\, 2\pi s w\left(\frac{\mu_0 s I}{2\pi a^2}\right)^2 = \frac{\mu_0 I^2 w}{4\pi a^4}\int_0^a s^3\, ds = \frac{\mu_0 I^2 w}{4\pi a^4}\frac{a^4}{4} = \frac{\mu_0 I^2 w}{16\pi}.$$

Total energy within gap is

$$\frac{I^2 t^2 w}{\epsilon_0 2\pi a^2} + \frac{\mu_0 I^2 w}{16\pi}.$$

Rate of increase of energy within gap is

$$\frac{I^2 t w}{\epsilon_0 \pi a^2}.$$

Poynting vector surface integrated over edge of gap is

$$S(2\pi a w) = \frac{I^2 a t}{\epsilon_0 2\pi^2 a^4}(2\pi a w) = \frac{I^2 t w}{\epsilon_0 \pi a^2}.$$