Mean separation

[1 point] According to *The Physics of QM* equations (15.17) and (15.18), the mean square separations $\langle (x_A - x_B)^2 \rangle$ are

for unsymmetrized wavefunctions (relevant to non-identical particles):
$$\langle x^2 \rangle_n + \langle x^2 \rangle_m - 2 \langle x \rangle_n \langle x \rangle_m$$
for symmetrized/antisymmetrized wavefunctions (relevant to identical particles):
the above $\mp 2 \left| \langle m|x|n \rangle \right|^2$

(Surprisingly, none of the integrals on the right involve integrands x_A or x_B, but simply x.)

[1 point] Thus we need to perform three integrals. Well, not really. It's obvious that $\langle x \rangle_n = L/2$, for all values of n.

[3 points] To find the mean of x^2 write

$$
\begin{aligned}
\langle x^2 \rangle_n &= \frac{2}{L} \int_0^L x^2 \sin^2 \left(\frac{n\pi}{L} x \right) dx \qquad \text{[use substitution } u = (n\pi/L)x \ldots \text{]} \\
&= \frac{2}{L} \left(\frac{L}{n\pi} \right)^3 \int_0^{n\pi} u^2 \sin^2 u \, du \qquad \text{[use Dwight equation 430.22 \ldots]} \\
&= \frac{2L^2}{n^3 \pi^3} \left[\frac{u^3}{6} - \left(\frac{u^2}{4} - \frac{1}{8} \right) \sin 2u - \frac{u}{4} \cos 2u \right]_0^{n\pi} \\
&= \frac{2L^2}{n^3 \pi^3} \left[\frac{n^3 \pi^3}{6} - \frac{n\pi}{4} \right] \\
&= L^2 \left[\frac{1}{3} - \frac{1}{2n^2 \pi^2} \right].
\end{aligned}
$$

Thus
$$
\langle x^2 \rangle_n + \langle x^2 \rangle_m - 2 \langle x \rangle_n \langle x \rangle_m = L^2 \left[\frac{1}{6} - \frac{1}{2\pi^2} \left(\frac{1}{n^2} + \frac{1}{m^2} \right) \right].
$$

[4 points] Meanwhile,

$$
\begin{aligned}
\langle m|x|n \rangle &= \frac{2}{L} \int_0^L x \sin \left(\frac{m\pi}{L} x \right) \sin \left(\frac{n\pi}{L} x \right) dx \qquad \text{[use substitution } \theta = (\pi/L)x \ldots \text{]} \\
&= \frac{2}{L} \left(\frac{L}{\pi} \right)^2 \int_0^\pi \theta \sin m\theta \sin n\theta \, d\theta \\
&= \frac{2L}{\pi^2} \int_0^\pi \theta \tfrac{1}{2} [\cos(n-m)\theta - \cos(n+m)\theta] \, d\theta \\
&= \frac{L}{\pi^2} \left[\int_0^\pi \theta \cos(n-m)\theta \, d\theta - \int_0^\pi \theta \cos(n+m)\theta \, d\theta \right].
\end{aligned}
$$

But for N an integer with $N \neq 0$,

$$
\begin{aligned}
\int_0^\pi \theta \cos N\theta \, d\theta &= \frac{1}{N^2} \int_0^{N\pi} u \cos u \, du \\
&= \frac{1}{N^2} \left[\cos u + u \sin u \right]_0^{N\pi} \\
&= \frac{1}{N^2} \left[(-1)^N - 1 \right] \\
&= \frac{1}{N^2} \begin{cases} -2 & \text{for } N \text{ odd} \\ 0 & \text{for } N \text{ even} \end{cases}
\end{aligned}
$$

So for $n - m$ even, $\langle m|x|n \rangle = 0$. But for $n - m$ odd

$$
\begin{aligned}
\langle m|x|n \rangle &= \frac{L}{\pi^2}(-2)\left[\frac{1}{(n-m)^2} - \frac{1}{(n+m)^2} \right] \\
&= -\frac{8L}{\pi^2} \frac{nm}{(n^2-m^2)^2}
\end{aligned}
$$

so the pivotal term is

$$
2|\langle m|x|n \rangle|^2 = \begin{cases} \dfrac{128\,L^2}{\pi^4} \dfrac{n^2 m^2}{(n^2-m^2)^4} & \text{for } n, m \text{ of opposite parity} \\[2mm] 0 & \text{for } n, m \text{ of same parity} \end{cases}
$$

⟦1 point⟧ In conclusion: For non-identical particles, or for identical particles in the case that n and m are of the same parity, the root-mean-square separation is

$$
\sqrt{\langle (x_A - x_B)^2 \rangle} = L \left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{m^2} \right) \right]^{1/2}
$$

while for symmetrized (minus sign) or antisymmetrized (plus sign) wavefunctions in the case that n and m are of opposite parity, the root-mean-square separation is

$$
\sqrt{\langle (x_A - x_B)^2 \rangle} = L \left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{m^2} \right) \mp \frac{128}{\pi^4} \frac{n^2 m^2}{(n^2-m^2)^4} \right]^{1/2}.
$$

There's a lot to explore in dissecting this result. Typically symmetrized means "huddle together" whereas antisymmetrized means "spread apart". But when n and m are of the same parity, then bosons, fermions, and non-identical particles all have the same root-mean-square separation. [For more non-typical cases, see D.F. Styer, "On the Separation of Identical Particles in Quantum Mechanics," *European Journal of Physics* **41** 065402 (7 pages) (14 October 2020).] Can you understand this any more deeply than just saying "it comes out of the math"? I can't.