Radiation

Griffiths problem 9.1: Matrix elements

It's clear from inspection that $\langle\eta|z|\eta\rangle = 0$ for all the traditional hydrogenic energy states. For the cross terms, use a table of spherical harmonics and a table of Coulomb problem wavefunctions. Remember that $z = r\cos\theta$. The four matrix elements desired are

$$\begin{aligned}
&\langle 1,0,0|z|2,0,0\rangle \\
&\langle 1,0,0|z|2,1,+1\rangle \implies \sim \int_0^{2\pi} d\phi\, e^{i\phi} = 0 \\
&\langle 1,0,0|z|2,1,0\rangle \\
&\langle 1,0,0|z|2,1,-1\rangle \implies \sim \int_0^{2\pi} d\phi\, e^{i\phi} = 0
\end{aligned}$$

Find the two remaining matrix elements using scaled units and $\mu = \cos\theta$:

$$\begin{aligned}
\langle 1,0,0|z|2,0,0\rangle &= \int_0^\infty r^2\,dr\,R_{10}(r)\,r\,R_{20}(r)\int_0^\pi \sin\theta\,d\theta\int_0^{2\pi} d\phi\,Y_0^{0*}(\theta,\phi)\,\cos\theta\,Y_0^0(\theta,\phi) \\
&= \int_0^\infty r^2\,dr\,R_{10}(r)\,r\,R_{20}(r)\underbrace{\int_{-1}^{+1} d\mu\,2\pi\sqrt{\frac{1}{4\pi}}\mu\sqrt{\frac{1}{4\pi}}}_{0} \\
&= 0.
\end{aligned}$$

Meanwhile

$$\begin{aligned}
\langle 1,0,0|z|2,1,0\rangle &= \int_0^\infty r^2\,dr\,R_{10}(r)\,r\,R_{21}(r)\int_0^\pi \sin\theta\,d\theta\int_0^{2\pi} d\phi\,Y_0^{0*}(\theta,\phi)\,\cos\theta\,Y_1^0(\theta,\phi) \\
&= \int_0^\infty r^3\,dr\,R_{10}(r)R_{20}(r)\int_{-1}^{+1} d\mu\,2\pi\left(\sqrt{\frac{1}{4\pi}}\right)\mu\left(\sqrt{\frac{3}{4\pi}}\mu\right).
\end{aligned}$$

The angular integral is

$$\int_{-1}^{+1} d\mu\,2\pi\left(\sqrt{\frac{1}{4\pi}}\right)\mu\left(\sqrt{\frac{3}{4\pi}}\mu\right) = \frac{\sqrt{3}}{2}\left[\frac{1}{3}\mu^3\right]_{-1}^{+1} = \frac{1}{\sqrt{3}}.$$

The radial integral is

$$\begin{aligned}
\int_0^\infty r^3\,dr\,R_{10}(r)R_{20}(r) &= \int_0^\infty r^3\,dr\,(2e^{-r})\left(\frac{1}{\sqrt{24}}re^{-r/2}\right) \\
&= \frac{1}{\sqrt{6}}\int_0^\infty r^4e^{-3r/2}\,dr \qquad [\![\text{use } u = 3r/2\ldots]\!] \\
&= \frac{1}{\sqrt{6}}\left(\frac{2}{3}\right)^5\int_0^\infty u^4e^{-u}\,du \\
&= \frac{1}{\sqrt{6}}\left(\frac{2}{3}\right)^5 4!.
\end{aligned}$$

So in conventional units

$$\langle 1,0,0|z|2,1,0\rangle = a_0 \left[\frac{1}{\sqrt{6}}\left(\frac{2}{3}\right)^5 4!\right]\left[\frac{1}{\sqrt{3}}\right] = a_0 \frac{2^7\sqrt{2}}{3^5} \approx 0.745\, a_0.$$

Griffiths problem 9.11: Decay times

General properties of lifetimes

From Griffiths, the lifetime is

$$\tau = \frac{1}{A} \qquad \text{where} \qquad A = \frac{\omega^3}{3\pi\epsilon_0 \hbar c^3}|\mathcal{P}|^2 = \frac{4}{3}\frac{\omega^3}{\hbar c^3}\frac{e^2}{4\pi\epsilon_0}|\langle b|\mathbf{r}|a\rangle|^2.$$

Our fist step is to convert to scaled units, using the dimensionless constant

$$\alpha = \frac{e^2}{4\pi\epsilon_0}\frac{1}{\hbar c} \cong \frac{1}{137}$$

as well as

$$a_0 = \frac{\hbar^2}{m(e^2/4\pi\epsilon_0)} \qquad \tau_0 = \frac{\hbar^3}{m(e^2/4\pi\epsilon_0)^2} \qquad \text{Ry} = \frac{1}{2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\frac{m}{\hbar^2}$$

or, equivalently,

$$\hbar = 2\,\text{Ry}\,\tau_0 \qquad m = 2\,\text{Ry}\,\frac{\tau_0^2}{a_0^2} \qquad \left(\frac{e^2}{4\pi\epsilon_0}\right) = 2\,\text{Ry}\,a_0.$$

This gives

$$\begin{aligned}
A &= \frac{4}{3}\frac{(\hbar\omega)^3}{\hbar}\alpha^3\left(\frac{4\pi\epsilon_0}{e^2}\right)^3\left(\frac{e^2}{4\pi\epsilon_0}\right)|\langle b|\mathbf{r}|a\rangle|^2 \\
&= \frac{4}{3}\alpha^3\frac{(\hbar\omega)^3}{\hbar}\left(\frac{4\pi\epsilon_0}{e^2}\right)^2|\langle b|\mathbf{r}|a\rangle|^2 \\
&= \frac{4}{3}\alpha^3\frac{(\hbar\omega)^3}{2\,\text{Ry}\,\tau_0}\frac{1}{4\,\text{Ry}^2 a_0^2}|\langle b|\mathbf{r}|a\rangle|^2 \\
\tau_0 A &= \frac{1}{6}\alpha^3\left(\frac{\hbar\omega}{\text{Ry}}\right)^3|\langle b|(\mathbf{r}/a_0)|a\rangle|^2.
\end{aligned}$$

This is my preferred expression for A. It applies to all states $|a\rangle$ and $|b\rangle$.

Our specific problem

Now, for our particular problem

$$\frac{\hbar\omega}{\text{Ry}} = \frac{\Delta E}{\text{Ry}} = \frac{(\,\text{Ry} - \frac{1}{4}\,\text{Ry}\,)}{\text{Ry}} = \frac{3}{4}$$

so

$$\tau_0 A = \frac{3^2}{2^7}\alpha^3|\langle b|(\mathbf{r}/a_0)|a\rangle|^2.$$

In scaled units

$$|\langle b|\mathbf{r}|a\rangle|^2 = |\langle b|x|a\rangle|^2 + |\langle b|y|a\rangle|^2 + |\langle b|z|a\rangle|^2.$$

We found the necessary z matrix elements in problem 9.1. Now for $x = r\sin\theta\cos\phi$ we need matrix elements like

$$\begin{aligned}
\langle 1,0,0|x|2,0,0\rangle &\implies \sim\int_0^{2\pi} d\phi\,\cos\phi = 0\\
\langle 1,0,0|x|2,1,+1\rangle &\\
\langle 1,0,0|x|2,1,0\rangle &\implies \sim\int_0^{2\pi} d\phi\,\cos\phi = 0\\
\langle 1,0,0|x|2,1,-1\rangle &
\end{aligned}$$

So we need to find

$$\begin{aligned}
\langle 1,0,0|x|2,1,\pm1\rangle &= \langle 1,0,0|r\sin\theta\cos\phi|2,1,\pm1\rangle\\
&= \int_0^\infty r^2\,dr\,R_{10}(r)\,r\,R_{21}(r)\int_0^\pi \sin\theta\,d\theta\,\sin\theta\int_0^{2\pi} d\phi\,Y_0^{0*}(\theta,\phi)\cos\phi Y_1^{\pm1}(\theta,\phi).
\end{aligned}$$

The radial integral we worked in problem 9.1: it is

$$\frac{1}{\sqrt{6}}\left(\frac{2}{3}\right)^5 4!$$

The angular integral is

$$\begin{aligned}
&\int_0^\pi \sin\theta\,d\theta\,\sin\theta\int_0^{2\pi} d\phi\,Y_0^{0*}(\theta,\phi)\cos\phi Y_1^{\pm1}(\theta,\phi)\\
&= \int_0^\pi \sin\theta\,d\theta\,\sin\theta\int_0^{2\pi} d\phi\sqrt{\frac{1}{4\pi}}\cos\phi\left(\mp\sqrt{\frac{3}{8\pi}}\right)\sin\theta e^{\pm i\phi}\\
&= \int_{-1}^{+1} d\mu\,\sqrt{1-\mu^2}\int_0^{2\pi} d\phi\sqrt{\frac{1}{4\pi}}\cos\phi\left(\mp\sqrt{\frac{3}{8\pi}}\right)\sqrt{1-\mu^2}\,e^{\pm i\phi}.
\end{aligned}$$

Now

$$\int_0^{2\pi} d\phi\,\cos\phi e^{\pm i\phi} = \int_0^{2\pi} d\phi\,\frac{e^{i\phi}+e^{-i\phi}}{2}e^{\pm i\phi} = \frac{1}{2}\int_0^{2\pi} d\phi\,(e^{\pm i2\phi}+1) = \pi$$

and

$$\int_{-1}^{+1} d\mu\,(1-\mu^2) = \left[\mu - \tfrac{1}{3}\mu^3\right]_{-1}^{+1} = \tfrac{4}{3},$$

whence

$$\langle 1,0,0|x|2,1,\pm1\rangle = \mp\frac{1}{\sqrt{6}}\left(\frac{2}{3}\right)^5 4!\pi\frac{4}{3}\sqrt{\frac{3}{2}}\frac{1}{4\pi} = \mp\frac{2^7}{3^5}.$$

Now use Griffiths [9.70] to find the y matrix elements:

$$\begin{aligned}
\langle n'\ell'm'|y|n\ell m\rangle &= i(m-m')\langle n'\ell'm'|x|n\ell m\rangle\\
\langle 1,0,0|y|2,0,0\rangle &= 0\\
\langle 1,0,0|y|2,1,0\rangle &= 0\\
\langle 1,0,0|y|2,1,\pm1\rangle &= i(\pm1)\langle 1,0,0|x|2,1,\pm1\rangle = -i\frac{2^7}{3^5}.
\end{aligned}$$

In summary

$$\begin{aligned}
|\langle 1,0,0|\mathbf{r}|2,0,0\rangle|^2 &= 0\\
|\langle 1,0,0|\mathbf{r}|2,1,0\rangle|^2 &= |\langle 1,0,0|z|2,1,0\rangle|^2 = \frac{2^{15}}{3^{10}}\\
|\langle 1,0,0|\mathbf{r}|2,1,\pm 1\rangle|^2 &= |\langle 1,0,0|x|2,1,\pm 1\rangle|^2 + |\langle 1,0,0|y|2,1,\pm 1\rangle|^2 = \frac{2^{15}}{3^{10}}.
\end{aligned}$$

Thus for the transition $|2,0,0\rangle \longrightarrow |1,0,0\rangle$, we have $A = 0$ so $\tau = \infty$.

Whereas for the transition $|2,1,m\rangle \longrightarrow |1,0,0\rangle$, we have

$$A = \frac{3^2}{2^7}\alpha^3\left(\frac{2^{15}}{3^{10}}\right) = \frac{2^8}{3^8}\alpha^3$$

so

$$\tau = \left(\frac{3}{2}\right)^8 \frac{1}{\alpha^3}\tau_0 \approx 6.59 \times 10^7\,\tau_0 = 1.60 \times 10^{-9}\text{ s.}$$

Remember from the first problem set that the time required for the innermost "Bohr orbit" is $2\pi\tau_0$, whence the decay time τ is 1.05×10^7 "orbital periods". If one "orbit" lasted as long as one heartbeat, then the decay time would last about four months.

A lifetime of 1.60×10^{-9} s is very short on a human time scale, but it corresponds to ten million "orbits" or, through the heartbeat-to-orbit analogy, to about one academic semester.

Griffiths problem 9.14

(a)

$$\begin{array}{ccccc}
 & \nearrow & |2,1,+1\rangle & \searrow & \\
|3,0,0\rangle & \longrightarrow & |2,1,0\rangle & \longrightarrow & |1,0,0\rangle\\
 & \searrow & |2,1,-1\rangle & \nearrow &
\end{array}$$

(b) The transition rates involve matrix elements like

$$|\langle 3,0,0|\mathbf{r}|2,1,m\rangle|^2$$

and it's clear from symmetry that these quantities are identical, so 1/3 decay through each channel. (These matrix elements are the same as those calculated in problem 9.11, except that every $R_{10}(r)$ must be replaced by an $R_{30}(r)$. Since $R_{10}(r)$ enters into only one integral, it would not be hard to do this problem quantitatively.)