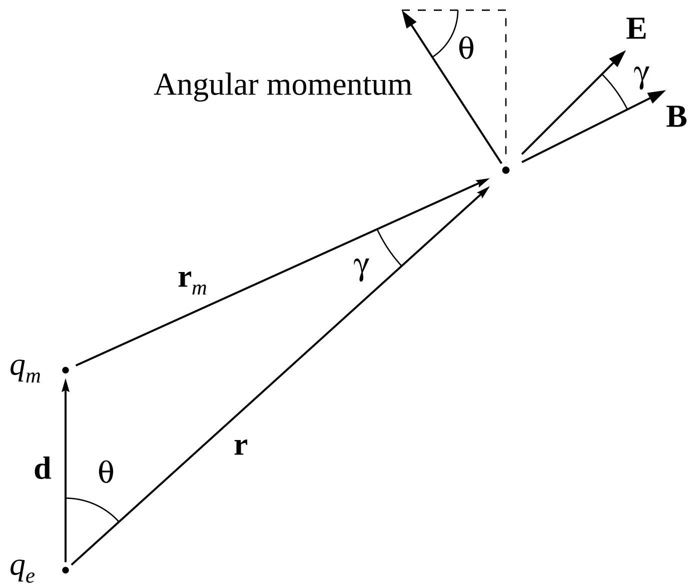
Thomson's dipole

Griffiths, *Electrodynamics*, fourth edition, problem 8.19



At the point in question:

Electric field is

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q_e}{r^2}\hat{r}.$$

Magnetic field is

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q_m}{r_m^2}\hat{r}_m.$$

Momentum density is

$$\vec{g}_{em} = \epsilon_0 \vec{E} \times \vec{B}$$

so it points into page with magnitude

$$|\vec{g}_{em}| = \frac{\mu_0}{(4\pi)^2}\frac{q_e q_m}{r^2 r_m^2}\sin\gamma.$$

Angular momentum density (origin at the electric charge) is

$$\vec{r} \times \vec{g}_{em}$$

but, by symmetry, the horizontal components of angular momentum will integrate to zero. Thus we want only the vertical component of angular momentum density at the point in question. This component is

$$\frac{\mu_0}{(4\pi)^2}\frac{q_e q_m}{r r_m^2}\sin\gamma\sin\theta.$$

⟦I think of the momentum as running around the axis in hoops — like hula hoops — centered on and perpendicular to the axis. Of course the angular momentum of each hoop is in the direction of $\vec{d}$. For no particular reason, I picture the momentum as yellow, and the hoops are like halos. So in my visualization the situation is somewhat angelic.⟧

First, we need to express r_m and γ in terms of d, r, and θ. According to the law of sines,

$$\frac{r_m}{\sin\theta} = \frac{d}{\sin\gamma}$$

so

$$\sin\gamma = \frac{d}{r_m}\sin\theta.$$

Meanwhile, according to the law of cosines,

$$r_m^2 = r^2 + d^2 - 2rd\cos\theta.$$

Thus the vertical component of the angular momentum density at the point in question is

$$\frac{\mu_0}{(4\pi)^2}\frac{q_e q_m}{r r_m^3} d\sin^2\theta = \frac{\mu_0}{(4\pi)^2}\frac{q_e q_m}{r[r^2 + d^2 - 2rd\cos\theta]^{3/2}} d\sin^2\theta.$$

Second, to find the total angular momentum, we need to integrate this density over all space. We use spherical coordinates. The direction of angular momentum is clearly from the electric charge toward the magnetic charge, so we need only find the magnitude

$$\begin{aligned} L &= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \int_0^{\infty} r^2\, dr\, \frac{\mu_0}{(4\pi)^2}\frac{q_e q_m}{r[r^2 + d^2 - 2rd\cos\theta]^{3/2}} d\sin^2\theta \\ &= \frac{\mu_0}{(4\pi)^2} q_e q_m\, 2\pi \int_0^{\pi} \sin\theta\, d\theta \int_0^{\infty} r^2\, dr\, \frac{1}{r[r^2 + d^2 - 2rd\cos\theta]^{3/2}} d\sin^2\theta \\ &= \frac{\mu_0}{8\pi} q_e q_m \int_0^{\pi} \sin^3\theta\, d\theta \int_0^{\infty} dr\, \frac{rd}{[r^2 + d^2 - 2rd\cos\theta]^{3/2}}. \end{aligned}$$

Change the integration variable from r to the dimensionless quantity $u = r/d$.

$$\begin{aligned} L &= \frac{\mu_0 q_e q_m}{8\pi}\int_0^{\pi} \sin^3\theta\, d\theta \int_0^{\infty} du\, \frac{ud^3}{[d^2u^2 + d^2 - 2d^2u\cos\theta]^{3/2}} \\ &= \frac{\mu_0 q_e q_m}{8\pi}\int_0^{\pi} \sin^3\theta\, d\theta \int_0^{\infty} du\, \frac{u}{[u^2 + 1 - 2u\cos\theta]^{3/2}}. \end{aligned}$$

This result is clearly independent of the magnitude d!

The radial integral is Dwight 380.013 giving

$$\begin{aligned} L &= \frac{\mu_0 q_e q_m}{8\pi}\int_0^{\pi} \sin^3\theta\, d\theta \left[-\frac{2(-2\cos\theta)u + 4}{(4 - 4\cos^2\theta)[u^2 + 1 - 2u\cos\theta]^{1/2}}\right]_0^{\infty} \\ &= \frac{\mu_0 q_e q_m}{8\pi}\int_0^{\pi} \sin^3\theta\, d\theta \left[\frac{u\cos\theta - 1}{\sin^2\theta[u^2 + 1 - 2u\cos\theta]^{1/2}}\right]_0^{\infty} \\ &= \frac{\mu_0 q_e q_m}{8\pi}\int_0^{\pi} \sin\theta\, d\theta\, [\cos\theta + 1] \end{aligned}$$

$$
\begin{aligned}
&= \frac{\mu_0 q_e q_m}{8\pi}\left[\int_0^\pi \sin\theta\cos\theta\,d\theta + \int_0^\pi \sin\theta\,d\theta\right] \\
&= \frac{\mu_0 q_e q_m}{8\pi}\left[0 - [\cos\theta]_0^\pi\right] \\
&= \frac{\mu_0 q_e q_m}{4\pi}.
\end{aligned}
$$

Appended question: Can you produce any sort of qualitative argument to understand why this angular momentum should be independent of d?

One approach is a very qualitative one: There is significant field angular momentum in the region where there is both significant $\vec{E}$ and significant $\vec{B}$. As the two monopoles draw apart, this region becomes larger but the fields become weaker

A second approach uses dimensional analysis.

quantity	dimensions (MKS)
ϵ_0	$\mathrm{C^2/N\,m^2 = m^{-3}\cdot kg^{-1}\cdot s^2\cdot C^2}$
μ_0	$\mathrm{N/A^2 = m\cdot kg\cdot C^{-2}}$
q_e	C
q_m	$\mathrm{C\,m/s = m\cdot s^{-1}\cdot C}$
d	m
L	$\mathrm{kg\,m^2/s = m^2\cdot kg\cdot s^{-1}}$

To make

$$L = (\text{dimensionless constant})\epsilon_0^\alpha \mu_0^\beta q_e^\gamma q_m^\delta d^\epsilon$$

we must have

$$
\begin{array}{ll}
\text{balancing m:} & 2 = -3\alpha + \beta + \delta + \epsilon \\
\text{balancing kg:} & 1 = -\alpha + \beta \\
\text{balancing s:} & -1 = 2\alpha - \delta \\
\text{balancing C:} & 0 = 2\alpha - 2\beta + \gamma + \delta
\end{array}
$$

There are four equations in five unknowns. The solutions are

$$
\begin{aligned}
\alpha &= \text{anything} \\
\beta &= 1 + \alpha \\
\gamma &= 1 - 2\alpha \\
\delta &= 1 + 2\alpha \\
\epsilon &= 0
\end{aligned}
$$

We know from above that the true solution happens to have $\alpha = 0$. But even without knowing this, the fact that $\epsilon = 0$ for any value of α means that the total angular momentum has to be independent of the separation d.

But my favorite approach is the one devised by David Lesser (class of 2010). He pointed out that the electric and magnetic monopoles exert no force on each other. Thus the two can be moved toward each other, or apart from each other, without the exertion of a force — or a torque. Moving them apart will thus require no torque and hence result in no change of angular momentum.

This argument is enticing, but I don't understand this aspect: "Similarly changing the orientation of the dipole will also require no force. Yet this change *does* alter the angular momentum!" So I'm not sure how all of this works out.

Appendix: Alternate Derivation

Gabe Salmon (class of 2018) suggested a different approach which is oriented more toward Cartesian coordinates. It has a number of strong points but neither Gabe nor I have been able to push it all the way through to a solution.

Set up coordinates with the electric charge at the origin, the magnetic charge at $\vec{d} = d\hat{z}$.

To begin with: By symmetry, the total angular momentum $\vec{L}$ must be along the axis between the two monopoles: $\vec{L} = +|\vec{L}|\hat{z}$ or $\vec{L} = -|\vec{L}|\hat{z}$. (This step is not absolutely necessary, but it's nice.)

The total angular momentum is

$$\vec{L} = \int_{\text{all space}} \vec{r} \times \vec{g}_{em}\, d^3r$$

but

$$\begin{aligned}
\vec{r} \times \vec{g}_{em} &= \epsilon_0 \vec{r} \times (\vec{E} \times \vec{B}) \\
&= \epsilon_0 \vec{r} \times \left(\left[\frac{1}{4\pi\epsilon_0} \frac{q_e}{|\vec{r}|^3} \vec{r} \right] \times \left[\frac{\mu_0}{4\pi} \frac{q_m}{|\vec{r} - \vec{d}|^3} (\vec{r} - \vec{d}) \right] \right) \\
&= \frac{\mu_0}{(4\pi)^2} \frac{q_e q_m}{|\vec{r}|^3 |\vec{r} - \vec{d}|^3} \vec{r} \times (\vec{r} \times (\vec{r} - \vec{d})) \\
&= -\frac{\mu_0}{(4\pi)^2} \frac{q_e q_m}{|\vec{r}|^3 |\vec{r} - \vec{d}|^3} \vec{r} \times (\vec{r} \times \vec{d})
\end{aligned}$$

so

$$\vec{L} = -\frac{\mu_0}{(4\pi)^2} q_e q_m \int \frac{\vec{r} \times (\vec{r} \times \vec{d})}{|\vec{r}|^3 |\vec{r} - \vec{d}|^3}\, d^3r.$$

Switching to dimensionless scaled variables

$$\tilde{\vec{r}} = \frac{\vec{r}}{|\vec{d}|} \qquad \text{and} \qquad \hat{z} = \frac{\vec{d}}{|\vec{d}|}$$

gives

$$\vec{L} = -\frac{\mu_0}{(4\pi)^2} q_e q_m \int \frac{d^3\, \tilde{\vec{r}} \times (\tilde{\vec{r}} \times \hat{z})}{d^6\, |\tilde{\vec{r}}|^3 |\tilde{\vec{r}} - \hat{z}|^3} (d^3)\, d^3\tilde{r} = -\frac{\mu_0}{(4\pi)^2} q_e q_m \int \frac{\tilde{\vec{r}} \times (\tilde{\vec{r}} \times \hat{z})}{|\tilde{\vec{r}}|^3 |\tilde{\vec{r}} - \hat{z}|^3}\, d^3\tilde{r}.$$

At this point, we know that $\vec{L}$ is independent of the separation magnitude d.

From now on, we will use only scaled variables so I will drop the tildes and write

$$\vec{L} = -\frac{\mu_0}{(4\pi)^2} q_e q_m \int \frac{\vec{r} \times (\vec{r} \times \hat{z})}{|\vec{r}|^3 |\vec{r} - \hat{z}|^3}\, d^3r.$$

From the well-known "bac cab" vector identity,

$$\vec{r} \times (\vec{r} \times \hat{z}) = \vec{r}\,(\vec{r} \cdot \hat{z}) - \hat{z}\,(\vec{r} \cdot \vec{r}) = \vec{r}\,z - \hat{z}\,r^2.$$

Look at the contribution from the $\vec{r}\,z$ part:

$$\int \frac{\vec{r}\,z}{|\vec{r}|^3 |\vec{r} - \hat{z}|^3}\,d^3r = \int \frac{(xz, yz, z^2)}{[x^2 + y^2 + z^2]^{3/2}[x^2 + y^2 + (z-1)^2]^{3/2}}\,d^3r.$$

Because the integrand is odd,

$$\int_{-\infty}^{+\infty} \frac{x}{[x^2 + A^2]^{3/2}[x^2 + B^2]^{3/2}}\,dx = 0,$$

and similarly for y, whence

$$\int \frac{\vec{r}\,z}{|\vec{r}|^3 |\vec{r} - \hat{z}|^3}\,d^3r = \hat{z} \int \frac{z^2}{|\vec{r}|^3 |\vec{r} - \hat{z}|^3}\,d^3r.$$

Pulling these strands together

$$\begin{aligned}
\vec{L} &= -\frac{\mu_0}{(4\pi)^2} q_e q_m \left[\hat{z} \int \frac{z^2}{|\vec{r}|^3 |\vec{r} - \hat{z}|^3}\,d^3r - \hat{z} \int \frac{r^2}{|\vec{r}|^3 |\vec{r} - \hat{z}|^3}\,d^3r\right] \\
&= \frac{\mu_0}{(4\pi)^2} q_e q_m \hat{z} \left[\int \frac{x^2 + y^2}{|\vec{r}|^3 |\vec{r} - \hat{z}|^3}\,d^3r\right].
\end{aligned}$$

The integral in square brackets is clearly a dimensionless positive number, so the angular momentum points from the electric monopole toward the magnetic monopole. This expression confirms the symmetry argument that we started with, gives us the direction of the angular momentum, its dependence on μ_0, q_e, and q_m, and its independence of d. All that remains is to find the dimensionless positive number

$$\begin{aligned}
\int \frac{x^2 + y^2}{|\vec{r}|^3 |\vec{r} - \hat{z}|^3}\,d^3r &= \int \frac{x^2 + y^2}{[x^2 + y^2 + z^2]^{3/2}[x^2 + y^2 + (z-1)^2]^{3/2}}\,d^3r \\
&= \int_{-\infty}^{+\infty} dz \int_0^{\infty} 2\pi s\,ds \frac{s^2}{[s^2 + z^2]^{3/2}[s^2 + (z-1)^2]^{3/2}} \\
&= \pi \int_{-\infty}^{+\infty} dz \int_0^{\infty} du \frac{u}{\{[u + z^2][u + (z-1)^2]\}^{3/2}},
\end{aligned}$$

but I haven't been able to evaluate this two-dimensional integral. Can you?