Ladder Operators for the Simple Harmonic Oscillator

a. Simple algeba shows that

$$\begin{aligned}\hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger)\\ \hat{p} &= -i\sqrt{\frac{m\hbar\omega}{2}}(\hat{a}-\hat{a}^\dagger)\end{aligned}$$

b. Matrix elements.

$$\begin{aligned}\langle m|\hat{a}|n\rangle &= \sqrt{n}\,\delta_{m,n-1}\\ \langle m|\hat{a}^\dagger|n\rangle &= \sqrt{n+1}\,\delta_{m,n+1}\\ \langle m|\hat{x}|n\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{m,n-1}+\sqrt{n+1}\,\delta_{m,n+1}\right)\\ \langle m|\hat{p}|n\rangle &= -i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{n}\,\delta_{m,n-1}-\sqrt{n+1}\,\delta_{m,n+1}\right)\end{aligned}$$

⟦**Question:** What's this about all the non-vanishing matrix elements of $\hat{p}$ being imaginary? I thought that $\hat{p}$ was Hermitian, and Hermiticity is associated with real matrix elements! **Answer:** Hermiticity requires that the diagonal matrix elements be real, not the off-diagonal. And the only non-vanishing matrix elements of $\hat{p}$ are off-diagonal.⟧

The next matrix elements could be obtained by expressing the relevant operators in terms of $\hat{a}$ and $\hat{a}^\dagger$, but I prefer the following device:

$$\begin{aligned}&\langle m|(\hat{a}\pm\hat{a}^\dagger)(\hat{a}\;{+/-}\;\hat{a}^\dagger)|n\rangle\\ =\;&\sum_\ell \langle m|(\hat{a}\pm\hat{a}^\dagger)|\ell\rangle\ell|(\hat{a}\;{+/-}\;\hat{a}^\dagger)|n\\ =\;&\sum_\ell \left(\sqrt{\ell}\,\delta_{m,\ell-1}\pm\sqrt{\ell+1}\,\delta_{m,\ell+1}\right)\left(\sqrt{n}\,\delta_{\ell,n-1}\;{+/-}\;\sqrt{n+1}\,\delta_{\ell,n+1}\right)\\ =\;&\sum_\ell \left[\sqrt{n}\,\delta_{\ell,n-1}\left(\sqrt{\ell}\,\delta_{m,\ell-1}\pm\sqrt{\ell+1}\,\delta_{m,\ell+1}\right)\;{+/-}\;\sqrt{n+1}\,\delta_{\ell,n+1}\left(\sqrt{\ell}\,\delta_{m,\ell-1}\pm\sqrt{\ell+1}\,\delta_{m,\ell+1}\right)\right]\\ =\;&\sqrt{n}\,\left(\sqrt{n-1}\,\delta_{m,n-2}\pm\sqrt{n}\,\delta_{m,n}\right)\;{+/-}\;\sqrt{n+1}\,\left(\sqrt{n+1}\,\delta_{m,n}\pm\sqrt{n+2}\,\delta_{m,n+2}\right)\\ =\;&\sqrt{n(n-1)}\,\delta_{m,n-2}+(\pm n\;{+/-}\;(n+1))\delta_{m,n}\;\times^{+}_{-}\;\sqrt{(n+1)(n+2)}\,\delta_{m,n+2}\end{aligned}$$

where the strange symbol $\times^{+}_{-}$ means $+$ if the signs in $\pm$ and ${+/-}$ are the same, and $-$ if they are different.

From this formula, we can read off

$$\begin{aligned}\langle m|\hat{x}^2|n\rangle &= \frac{\hbar}{2m\omega}\left[\sqrt{n(n-1)}\,\delta_{m,n-2}+(2n+1)\delta_{m,n}+\sqrt{(n+1)(n+2)}\,\delta_{m,n+2}\right]\\ \langle m|\hat{p}^2|n\rangle &= -\frac{m\hbar\omega}{2}\left[\sqrt{n(n-1)}\,\delta_{m,n-2}-(2n+1)\delta_{m,n}+\sqrt{(n+1)(n+2)}\,\delta_{m,n+2}\right]\\ \langle m|\hat{x}\hat{p}|n\rangle &= -i\frac{\hbar}{2}\left[\sqrt{n(n-1)}\,\delta_{m,n-2}-\delta_{m,n}-\sqrt{(n+1)(n+2)}\,\delta_{m,n+2}\right]\\ \langle m|\hat{p}\hat{x}|n\rangle &= -i\frac{\hbar}{2}\left[\sqrt{n(n-1)}\,\delta_{m,n-2}+\delta_{m,n}-\sqrt{(n+1)(n+2)}\,\delta_{m,n+2}\right]\end{aligned}$$

(Note: The fourth equation follows directly from the third because $[\hat{x}, \hat{p}] = i\hbar$ implies $\langle m|\hat{x}\hat{p}|n\rangle = \langle m|\hat{p}\hat{x}|n\rangle + i\hbar\delta_{m,n}$.)

Finally, becuase $\hat{H}|n\rangle = \hbar\omega(n + \frac{1}{2})|n\rangle$,

$$\langle m|\hat{H}|n\rangle = \hbar\omega(n + \tfrac{1}{2})\delta_{m,n}.$$

c. Kinetic and potential energy expectation values in an energy state.

$$\langle\hat{T}\rangle = \langle n|\hat{T}|n\rangle = \frac{1}{2m}\langle n|\hat{p}^2|n\rangle = \frac{1}{2m}\frac{m\hbar\omega}{2}(2n+1) = \frac{\hbar\omega}{2}(n + \tfrac{1}{2}).$$

Because

$$\begin{aligned}
\hat{H} &= \hat{T} + \hat{U}, \\
\langle\hat{H}\rangle &= \langle\hat{T}\rangle + \langle\hat{U}\rangle, \\
\langle\hat{U}\rangle &= \langle\hat{H}\rangle - \langle\hat{T}\rangle \\
&= \hbar\omega(n + \tfrac{1}{2}) - \frac{\hbar\omega}{2}(n + \tfrac{1}{2}) \\
&= \frac{\hbar\omega}{2}(n + \tfrac{1}{2}).
\end{aligned}$$

d. Uncertainties for an energy state.

In any energy state, $\langle\hat{x}\rangle = 0$ and $\langle\hat{p}\rangle = 0$, so

$$\begin{aligned}
(\Delta x)^2 &= \langle\hat{x}^2\rangle = \frac{\hbar}{2m\omega}(2n+1) \\
(\Delta p)^2 &= \langle\hat{p}^2\rangle = \frac{m\hbar\omega}{2}(2n+1)
\end{aligned}$$

Thus

$$\begin{aligned}
\Delta x &= \sqrt{\frac{\hbar}{m\omega}}\sqrt{n + \tfrac{1}{2}} \\
\Delta p &= \sqrt{m\hbar\omega}\sqrt{n + \tfrac{1}{2}} \\
\Delta x\Delta p &= \hbar(n + \tfrac{1}{2}).
\end{aligned}$$

So the SHO ground state is a minimum uncertainty state.