The rotating water glass

a. Consider a cube of water of volume $\ell \times \ell \times \ell$ where $\ell$ is “quite small”.

The forces due to pressure are denoted $F_{t,1}$ (force tangential, 1) and so forth.

Use $\sum \vec{F} = m \vec{a}$:

**Tangential component:**

$$F_{t,1} - F_{t,2} = ma_t = 0$$

$$\implies p_{t,1} \ell^2 = p_{t,2} \ell^2 \implies p_{t,1} = p_{t,2}$$

Thus the pressure is independent of angle (which is also clear from symmetry).

**Vertical component:**

$$F_{v,1} - \rho Vg - F_{v,2} = ma_v = 0$$

$$\implies p_{v,1} \ell^2 - \rho \ell^3 g - p_{v,2} \ell^2 = 0 \implies p_{v,2} - p_{v,1} = -\rho \ell g$$

Thus $\frac{\partial p}{\partial h} = -\rho g$.

**Radial component:**

$$F_{r,1} - F_{r,2} = ma_r = -\rho V \omega^2 r$$  (centripetal acceleration)

$$\implies p(r) \ell^2 - p(r + \ell) \ell^2 = -\rho \ell^3 \omega^2 r$$

Thus $\frac{\partial p}{\partial r} = \rho \omega^2 r$. 

1
b. When moving along a small path, the change in pressure is

\[ dp = \frac{\partial p}{\partial r} \, dr + \frac{\partial p}{\partial h} \, dh = \rho \omega^2 r \, dr - \rho g \, dh. \]

Integrate this differential along the following integration path

\[ \int dp = \int_0^r \rho \omega^2 r \, dr - \int_{h_c}^h \rho g \, dh \]

\[ \int dp = \int_0^r \rho \omega^2 r \, dr - \int_{h_c}^h \rho g \, dh \]

\[ p - p_a = \frac{1}{2} \rho \omega^2 r^2 - \rho g (h - h_c). \]

c. At the fluid surface, \( p = p_a \), so

\[ 0 = \frac{1}{2} \rho \omega^2 r^2 - \rho g (h - h_c) = \frac{1}{2} \rho \omega^2 r^2 - \rho g (y(r)) \]

whence

\[ y(r) = \frac{\omega^2}{2g} r^2. \]

Note that the profile \( y(r) \) is independent of \( \rho \): water and mercury will have identical surface profiles.