# The Klein-Gordon Propagator

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This document fills in some of the details behind the discussion of amplitudes to go from place to place found in Richard Feynman's *QED: The Strange Theory of Light and Matter* (Princeton University Press, Princeton, New Jersey, 1985). Unlike Feynman's book, this document is technical. To understand it, you need to understand terms like "contour integral" and "residue".

The discussion I wish to elucidate is found on pages 87–91 of *QED*, and presents amplitudes for the first two of the three basic actions, namely "a photon goes from place to place" and "an electron goes from place to place".

Feynman's so-called "polarization-free photon" and "spin-zero electron" are technically called "Klein-Gordon particles" of zero and finite mass, respectively. The "amplitude to go from place to place" that Feynman mentions is called the "Klein-Gordon propagator". An integral expression for this propagator is given in, for example, Claude Itzykson and Jean-Bernard Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980), page 35, equation (1-178), or in Kurt Gottfried and Victor Weisskopf, *Concepts of Particle Physics* (Oxford University Press, New York, 1986), volume II, page 230, equation (48) [note misprint: $d^4x$ should read $d^4Q$]. Using the phase conventions of *QED*, the propagator to change space-time position by $x = (c\Delta t, \Delta\mathbf{r}) = (x_0, \mathbf{x})$ is

$$G_F(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot x} \frac{1}{p^2 - m^2 + i\epsilon}, \tag{1}$$

where $x \cdot y \equiv x_0 y_0 - \mathbf{x}\cdot\mathbf{y}$. The aim of this document is to show how the qualitative amplitude descriptions of *QED* follow from this expression.

## 1 The energy integral

Write the propagator as

$$G_F(x) = \int \frac{d^3p}{(2\pi)^4}\, e^{-i\mathbf{p}\cdot\mathbf{x}} \int dp_0\, e^{ip_0 x_0} \frac{1}{p_0^2 - \mathbf{p}^2 - m^2 + i\epsilon}. \tag{2}$$

Define $E = +\sqrt{\mathbf{p}^2 + m^2}$, and then evaluate the energy integral

$$I \equiv \int dp_0 \, e^{ip_0 x_0} \frac{1}{p_0^2 - E^2 + i\epsilon}, \tag{3}$$

using contour integration in the complex $p_0$ plane. To locate the poles, write

$$p_0^2 - E^2 + i\epsilon = p_0^2 - \left(E - \frac{i\epsilon}{2E}\right)^2 = \left[p_0 + \left(E - \frac{i\epsilon}{2E}\right)\right]\left[p_0 - \left(E - \frac{i\epsilon}{2E}\right)\right]. \tag{4}$$

Thus there are two poles: one just above the real axis and one just below. The first pole has

$$\text{location:} \quad -\left(E - \frac{i\epsilon}{2E}\right) \qquad \text{residue:} \quad -\frac{\exp\{-i\,(E - i\epsilon/2E)\,x_0\}}{2\,(E - i\epsilon/2E)}, \tag{5}$$

while the second has

$$\text{location:} \quad +\left(E - \frac{i\epsilon}{2E}\right) \qquad \text{residue:} \quad +\frac{\exp\{+i\,(E - i\epsilon/2E)\,x_0\}}{2\,(E - i\epsilon/2E)}. \tag{6}$$

If $x_0 > 0$, we close the contour on the top half plane enclosing the first pole to find (in the limit $\epsilon \to 0$)

$$I = +2\pi i \left(-\frac{e^{-iEx_0}}{2E}\right), \tag{7}$$

while if $x_0 < 0$, we close the contour on the bottom half plane enclosing the second pole to find

$$I = -2\pi i \left(+\frac{e^{+iEx_0}}{2E}\right). \tag{8}$$

These two expressions can be written as one,

$$I = -2\pi i \frac{e^{-iE|x_0|}}{2E}, \tag{9}$$

whence we conclude

$$G_F(x) = -\frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \, e^{-i\mathbf{p}\cdot\mathbf{x}} \frac{e^{-iE|x_0|}}{E}, \qquad \text{where} \qquad E = +\sqrt{p^2 + m^2}\,. \tag{10}$$

## 2 Propagator for massless particles

If $m = 0$, then $E = |\mathbf{p}|$ and the above expression becomes

$$G_F(x) = -\frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \, e^{-i\mathbf{p}\cdot\mathbf{x}} \frac{e^{-i|\mathbf{p}||x_0|}}{|\mathbf{p}|}. \tag{11}$$

For the case $x_0 \neq 0$, this integral is evaluated in the appendix and is

$$G_F(x) = -\frac{i}{(2\pi)^2} \frac{1}{\mathbf{x}^2 - x_0^2} = -\frac{i}{(2\pi)^2} \frac{1}{(\Delta\mathbf{r})^2 - (c\Delta t)^2}. \tag{12}$$

Thus

$$\begin{array}{llll} \text{if} & (\Delta\mathbf{r})^2 > (c\Delta t)^2 & (\text{i.e. } v > c) & \text{then } G_F \sim -i \\ \text{if} & (\Delta\mathbf{r})^2 < (c\Delta t)^2 & (\text{i.e. } v < c) & \text{then } G_F \sim +i \end{array}$$

These amplitudes correspond to the two little arrows pointing to the right and to the left in figure 56 on page 90 of *QED*.

The remaining case is $(\Delta\mathbf{r})^2 = (c\Delta t)^2$, that is $v = c$. In this case $x_0^2 = \mathbf{x}^2$ and

$$G_F(x) = -\frac{i}{2}\int \frac{d^3p}{(2\pi)^3}\, e^{-i\mathbf{p}\cdot\mathbf{x}}\frac{e^{-i|\mathbf{p}||\mathbf{x}|}}{|\mathbf{p}|}. \tag{13}$$

Convert this integral into spherical coordinates (using $\mu = \cos(\theta)$) to find

$$G_F(x) = -\frac{i}{2(2\pi)^3}2\pi\int_0^\infty p^2\,dp\int_{-1}^{+1} d\mu\, e^{-ip|\mathbf{x}|\mu}\frac{e^{-ip|\mathbf{x}|}}{p}. \tag{14}$$

The integral over $\mu$ is

$$\int_{-1}^{+1} d\mu\, e^{-ip|\mathbf{x}|\mu} = \frac{2\sin(p|\mathbf{x}|)}{p|\mathbf{x}|}, \tag{15}$$

so

$$\begin{aligned} G_F(x) &= -\frac{i}{(2\pi)^2}\frac{1}{|\mathbf{x}|}\int_0^\infty dp\,\sin(p|\mathbf{x}|)e^{-ip|\mathbf{x}|} \\ &= -\frac{i}{(2\pi)^2}\frac{1}{|\mathbf{x}|^2}\int_0^\infty du\,\sin(u)(\cos(u) - i\sin(u)). \end{aligned} \tag{16}$$

Now, the integral

$$\int_0^\infty du\,\sin(u)\cos(u) \quad \text{is bounded,} \tag{17}$$

whereas

$$\int_0^\infty du\,\sin^2(u) \quad \text{approaches infinity.} \tag{18}$$

Thus for the remaining case $v = c$, we have

$$G_F(x) = -\frac{(\text{real positive infinity})}{|\mathbf{x}|^2}. \tag{19}$$

This amplitude corresponds to the big arrow pointing straight up in figure 56 on page 90 of *QED*.

## Appendix: Fourier transform of the Yukawa Potential

**Theorem:** If

$$f(\mathbf{r}) = \frac{e^{-k_0 r}}{r} \quad \text{with} \quad k_0 > 0, \tag{20}$$

and if

$$\tilde{f}(\mathbf{k}) = \int d^3r\, e^{-i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r}) \tag{21}$$

$$f(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3}\, e^{+i\mathbf{k}\cdot\mathbf{r}}\tilde{f}(\mathbf{k}), \tag{22}$$

then

$$\tilde{f}(\mathbf{k}) = \frac{4\pi}{k^2 + k_0^2}. \tag{23}$$

**Proof:**

$$\begin{aligned}
\tilde{f}(\mathbf{k}) &= \int d^3r \, e^{-i\mathbf{k}\cdot\mathbf{r}} \frac{e^{-k_0 r}}{r} & (24)\\
&= 2\pi \int_0^\infty r^2 \, dr \left( \int_{-1}^{+1} d\mu \, e^{-ikr\mu} \right) \frac{e^{-k_0 r}}{r} \qquad \text{(where } \mu = \cos\theta\text{)} & (25)\\
&= 2\pi \int_0^\infty r^2 \, dr \left( \frac{2\sin(kr)}{kr} \right) \frac{e^{-k_0 r}}{r} & (26)\\
&= \frac{4\pi}{k} \int_0^\infty dr \, \sin(kr) e^{-k_0 r} & (27)\\
&= \frac{4\pi}{k} \Im m \left\{ \int_0^\infty dr \, e^{(ik-k_0)r} \right\} & (28)\\
&= \frac{4\pi}{k} \Im m \left\{ \frac{e^{(ik-k_0)r}}{ik - k_0} \right\}_{r=0}^{\infty} & (29)\\
&= \frac{4\pi}{k^2 + k_0^2} & (30)
\end{aligned}$$

# 3 Propagator for massive particles

Returning to the very beginning of this discussion,

$$G_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \, e^{ip\cdot(x-y)} \frac{1}{p^2 - m^2 + i\epsilon}. \tag{31}$$

But, recognizing the geometric series,

$$\begin{aligned}
\frac{1}{p^2 - m^2 + i\epsilon} &= \frac{1}{(p^2 + i\epsilon)[1 - m^2/(p^2 + i\epsilon)]} \\
&= \frac{1}{(p^2 + i\epsilon)} \left[ 1 + \frac{m^2}{p^2 + i\epsilon} + \frac{(m^2)^2}{(p^2 + i\epsilon)^2} + \frac{(m^2)^3}{(p^2 + i\epsilon)^3} + \cdots \right].
\end{aligned}$$

Therefore

$$G_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{p^2 + i\epsilon} + m^2 \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{(p^2 + i\epsilon)^2} + (m^2)^2 \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{(p^2 + i\epsilon)^3} + \cdots. \tag{32}$$

Note that the first term in this series is nothing but the zero-mass propagator, which we will call

$$G_F^{(0)}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{p^2 + i\epsilon}. \tag{33}$$

I'm going to write the second integral in a funny way, using the four-dimensional Dirac delta function $\delta^{(4)}(p)$:

$$\int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{(p^2 + i\epsilon)^2} = \int d^4p' \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot x} e^{-ip'\cdot y}}{(p^2 + i\epsilon)(p'^2 + i\epsilon)} \delta^{(4)}(p' - p). \tag{34}$$

Use the integral expression

$$\delta^{(4)}(p'-p) = \int \frac{d^4x'}{(2\pi)^4}\, e^{i(p'-p)\cdot x'} \tag{35}$$

for that delta function to write

$$\int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{(p^2+i\epsilon)^2} = \int d^4x' \int \frac{d^4p'}{(2\pi)^4} \frac{e^{ip'\cdot(x'-y)}}{(p'^2+i\epsilon)} \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x-x')}}{(p^2+i\epsilon)}. \tag{36}$$

Recognizing the two zero-mass propagators on the right, we write

$$\int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{(p^2+i\epsilon)^2} = \int d^4x'\, G_F^{(0)}(x-x')G_F^{(0)}(x'-y). \tag{37}$$

In a similar way, the third integral in the series (the one multiplying $(m^2)^2$) can be written as a double integral of a product of three zero-mass propagators, and so forth. We conclude that

$$\begin{aligned} G_F(x-y) &= G_F^{(0)}(x-y) \\ &\quad + m^2 \int d^4x'\, G_F^{(0)}(x-x')G_F^{(0)}(x'-y) \\ &\quad + (m^2)^2 \int d^4x' \int d^4x''\, G_F^{(0)}(x-x')G_F^{(0)}(x'-x'')G_F^{(0)}(x''-y) \\ &\quad + \cdots . \end{aligned} \tag{38}$$

This is precisely the "boxes within boxes" form described in footnote 3 on page 91 of *QED*.

Note that the analysis of this section links the finite-mass propagator to the zero-mass propagator without ever using the previously-obtained explicit form of the zero-mass propagator.