Calculation of the anomalous magnetic moment of the electron

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Richard Feynman's little book *QED: The Strange Theory of Light and Matter* (Princeton University Press, 1985) discusses the anomalous magnetic moment of the electron, $(g-2)/2 = \mu/\mu_0 - 1$, on pages 115–118. This document gives some historical references and discusses work after the publication of *QED*.

The term linear in α (or, as Feynman likes to call it, "the term with two extra j's") was found by Julian Schwinger in 1947 to be

$$\frac{\alpha}{2\pi}.$$

This involves one Feynman diagram. Julian Schwinger, "On quantum-electrodynamics and the magnetic moment of the electron," Physical Review, 73 (1948) 416–417 [received 30 December 1947] and "Quantum electrodynamics III: The electromagnetic properties of the electron—radiative corrections to scattering," Physical Review, 76 (1949) 790–817.

Next the term quadratic in α was found, involving 7 diagrams. (Four representative diagrams are shown in figure 75 of *QED*.) On page 117, Feynman tells the story:

> It took two "independent" groups of physicists two years to calculate this next term [the one with four extra couplings], and then another year to find out there was a mistake — experimenters had measured the value to be slightly different, and it looked for awhile that the theory didn't agree with experiment for the first time, but no: it was a mistake in arithmetic. How could two groups make the same mistake? It turns out that near the end of the calculation the two groups compared notes and ironed out the differences between their calculations, so they were not really independent.

The facts differ slightly from this story. The initial calculation was performed not by two groups but by two individuals, Robert Karplus and Norman M. Kroll, who were colleagues as postdoctoral fellows at the Institute for Advanced Study, in the academic year 1948/49 — the calculation took less than a year. The error was discovered by André Petermann eight years later.

The initial calculation was reported in Robert Karplus and Norman M. Kroll, "Fourth-order corrections in quantum electrodynamics and the magnetic moment of the electron," Physical Review, 77 (1950) 536–549 [received 17 October 1949]. Karplus and Kroll found a value of

$$\frac{\alpha}{2\pi} + \left[-\frac{2687}{288} + \frac{125}{36}\pi^2 - 9\pi^2 \ln 2 + 28\,\zeta(3)\right]\left(\frac{\alpha}{\pi}\right)^2 + \cdots \approx \frac{\alpha}{2\pi} - 2.973\left(\frac{\alpha}{\pi}\right)^2.$$

[Here ζ is the Riemann zeta function.] Although Karplus and Kroll were colleagues, they claim (footnote 23 of their paper) that "two independent calculations ... were performed so as to provide some check of the final result." [Note typo in KK paper: in equation (62a), first "3" should be "2". Note also that figure 1 in this paper shows 18 diagrams, not 7. However the discussion on page 539 shows that only 7 diagrams contribute to the magnetic moment.]

The error was uncovered by A. Petermann, "Fourth order magnetic moment of the electron," Helvetica Physica Acta, 30 (1957) 407–408 [received 17 August 1957]. (See also A. Petermann, "Magnetic moment of the μ meson," Physical Review, 105 (1957) 1931.) And also by Charles M. Sommerfield, "The magnetic moment of the electron," Annals of Physics, 5 (1958) 26–57. These scientists found a value of

$$\frac{\alpha}{2\pi} + \left[\frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3)\right]\left(\frac{\alpha}{\pi}\right)^2 + \cdots \approx \frac{\alpha}{2\pi} - 0.328\left(\frac{\alpha}{\pi}\right)^2.$$

Norman Kroll tells his version of these events in an interview with Finn Aaserud: (Interview conducted on 28 June 1986, Niels Bohr Library & Archives, American Institute of Physics, College Park, MD, USA, `http://www.aip.org/history/ohilist/28394.html`.)

> **Kroll:** ... that was the year [1948/49] that Dyson showed how to do higher order perturbation theory and how to do normalization in a consistent way. Karplus and I carried out the first major application of that program, to calculate the fourth order magnetic moment, which calculation subsequently turned out to have some errors in it, which has been a perpetual source of embarrassment to me, but nevertheless the paper I believe was quite influential. First of all, it was quite some time before the errors were found, and during that time the illustration of the way you actually do a renormalized calculation — the demonstration of the fact you can actually do all those integrals and get an answer — was I think influential. Anyhow, that's what we did that year.
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> **Aaserud:** That kind of mistake might just indicate the newness of the discovery.
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> **Kroll:** They were arithmetic, as a matter of fact. I would say the thing that I learned from that is in doing a complicated calculation, you have to take the same kinds of precautions that an experimenter takes to see that dirt doesn't get in his apparatus. We had some internal checks but not nearly enough.
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> **Aaserud:** But it got published; it was refereed, wasn't it?
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> **Kroll:** Oh yes, it was refereed and published and was a famous paper and now it's an infamous paper.

This is not the end of the saga. The term proportional to α^3, “the term with *six* extra j’s,” requires 72 diagrams. (Three representative diagrams are shown in figure 76 of *QED*.) The value was first found numerically, but then in 1996 an equation for it was discovered! [S. Laporta and E. Remiddi, “The analytical value of the electron $(g-2)$ at order α^3 in QED,” Physics Letters B, 379 (1996) 283–291.] Just for fun, I will write down the equation:

$$\left[\tfrac{28259}{5184}+\tfrac{17101}{810}\pi^2-\tfrac{298}{9}\pi^2\ln 2+\tfrac{139}{18}\zeta(3)-\tfrac{239}{2160}\pi^4-\tfrac{100}{72}\pi^2\ln^2 2+\tfrac{100}{72}\ln^4 2+\tfrac{100}{3}a_4-\tfrac{215}{24}\zeta(5)+\tfrac{83}{72}\pi^2\zeta(3)\right]\left(\tfrac{\alpha}{\pi}\right)^3.$$

where $a_4=\sum_{n=1}^{\infty}2^{-n}n^{-4}$. It takes a lot of work just to write down and pronounce this result, so you can imagine how much work was involved in calculating it! This result came out eight years after Feynman’s death, but I can imagine him looking at the equation with a sly smile.

Feynman says “the term for eight extra j’s involves something like nine hundred diagrams [actually 891], with a hundred thousand terms each — a fantastic calculation — and it’s being done right now.” These terms are evaluated numerically, and the most recent version of this result is T. Aoyama, M. Hayakawa, T. Kinoshita, and M. Nio, “Revised value of the eighth-order QED contribution to the anomalous magnetic moment of the electron,” Physical Review D, 77 (2008) 053012.

Will scientists stop at this point? Of course not. Right now progress is being made on the 12672 diagrams involving ten couplings. Tatsumi Aoyama, Masashi Hayakawa, Toichiro Kinoshita, and Makiko Nio, “Tenth-order QED lepton anomalous magnetic moment: Eighth-order vertices containing a second-order vacuum polarization,” Physical Review D 85 (2012) 033007. This paper gives a good overview of the situation today: The best experimental value is

$$1.001\,159\,652\,180\,73\ (\pm 28),$$

while the best theoretical value is

$$1.001\,159\,652\,181\,13\ (\pm 86).$$

Feynman wrote that the accuracy available when he wrote in 1985 was “the equivalent of measuring the distance from Los Angeles to New York, a distance of over 3,000 miles, to within the width of a human hair.” This is a bit of an overstatement: that accuracy was the equivalent of measuring the distance from Los Angeles to New York to within a millimeter — ten human hairs. The accuracy available today is more than 600 times better — it is the equivalent of measuring the distance from Los Angeles to New York to within the width of a bacterium, or of measuring the distance from the Earth to the Moon to within the width of a single human hair.