Quantum revivals versus classical periodicity in the infinite square well

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Abstract

A particle of mass M moves in an infinite square well of width L (the "particle in a box"). Classically, the motion has period $L\sqrt{2M/E}$, which depends upon the initial condition through the energy E. Quantum mechanically, any wavefunction repeats exactly with period $4ML^2/\pi\hbar$, independent of the initial condition. Given this qualitative difference, how can the classical motion possibly be the limit of the quantal time development? The resolution of this paradox involves the difference between the exact revival (recurrence) of the wavefunction and the approximate periodicity of expectation values such as $\langle x(t)\rangle$. (The latter may recur an odd integral number of times before the full wavefunction recurs.) The period of the expectation values *does* depend on the initial condition and *can* possess the expected classical limit.

The classical limit of quantal time development, first studied by Ehrenfest [1], is still an area of active investigation [2, 3]. Much of this research centers on systems with two or more degrees of freedom, where classical chaos enters the picture [4, 5], and much centers on approximate quantal revivals in complicated situations [6, 7, 8]. Yet this article will show that even that workhorse of introductory quantum mechanics, the one-dimensional infinite square well, exhibits surprising features, the elucidation of which contributes to one's intuition concerning quantum mechanics and illuminates the more general problems.

The topic of quantal time development in the infinite square well has been treated theoretically by Bluhm, Kostelecký, and Porter [9], and by Aronstein and Stroud [10]. In these treatments the results flow ultimately either from the WKB approximation [11] or from a sophisticated Taylor series expansion of the eigenenergies E_n as a function of the integer quantum number n. These approaches produce results of great generality.

In contrast, the approach of this paper applies only to the infinite square well, yet this restriction permits a far more direct and concrete argument — indeed, the calculations in the body of this paper could be carried out by a student a few weeks into a standard Junior or Senior level quantum mechanics course.

A decade ago, the topic of this paper would have been of great but purely theoretical interest. Contemporary developments in quantum-well nanostructures [12] and in femtosecond pulsed-laser excitation [13] have, however, rendered these results potentially testable in the laboratory. Indeed, experiments of this type have already been performed [14, 15].

I. The situation

A particle of mass M moves in an infinite square well of width L, centered on the origin. This system has energy eigenvalues

$$E_n = \frac{\pi^2\hbar^2}{2ML^2}n^2 = E_1 n^2 \quad \text{for} \quad n = 1, 2, 3, \ldots. \tag{1}$$

Notice that as the energy increases, the separation between energy eigenvalues *increases* rather than decreasing to a continuum as is usual, and as is expected for the classical limit [16]. Thus one is entitled to suspect some pathology concerning quantal time development in the classical limit. The aim of this article is to investigate such suspicions.

For future reference, we present the energy eigenfunctions [17]

$$\eta_n(x) = \sqrt{\frac{2}{L}}\cos\left(\frac{\pi}{L}nx\right) \quad \text{for} \quad n = 1, 3, 5, \ldots \tag{2}$$

$$\eta_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{L}nx\right) \quad \text{for} \quad n = 2, 4, 6, \ldots \tag{3}$$

and the matrix elements (obtained through straightforward integration)

$$\langle m|x|n\rangle = \begin{cases} 0 & \text{if } \eta_n(x) \text{ and } \eta_m(x) \text{ are of the same parity,} \\ \dfrac{8L}{\pi^2}(-1)^{(m+n+1)/2}\dfrac{mn}{(m^2-n^2)^2} & \text{if } \eta_n(x) \text{ and } \eta_m(x) \text{ are of opposite parity.} \end{cases} \tag{4}$$

II. The revival paradox

Classical periodicity. It is a trivial matter to show that a classical particle of energy E in the infinite square well bounces back and forth between the walls with period

$$T_{\mathrm{cl}} = L\sqrt{2M/E}. \tag{5}$$

Wavefunction revival. Suppose the initial wavefunction is

$$\psi(x;0) = \sum_{n=1}^{\infty} c_n \eta_n(x). \tag{6}$$

(Some of the expansion coefficients c_n may of course vanish.) This wavefunction evolves in time to

$$\psi(x;t) = \sum_{n=1}^{\infty} c_n e^{-iE_n t/\hbar} \eta_n(x). \tag{7}$$

Does it ever happen that at some revival time $T_{\rm rev}$ the quantal state represented by $\psi(x;T_{\rm rev})$ is exactly the same as the initial state $\psi(x;0)$?

Such a revival will happen whenever all of the phase factors $e^{-iE_n t/\hbar}$ are equal. (Note that the phase factors do not all have to all equal one, because the two wavefunctions $\psi(x)$ and $e^{-i\varphi}\psi(x)$ represent the same quantal state.) That is, it happens whenever

$$\frac{E_n}{\hbar} T_{\rm rev} = 2\pi N_n + \varphi \quad \text{ for all } n \text{ such that } c_n \neq 0. \tag{8}$$

Here N_n is an integer (positive, negative, or zero) that can vary from one value of n to another, whereas the time $T_{\rm rev}$ and the phase φ *must not* vary with n.

Use of the eigenenergy result $E_n = E_1 n^2$ gives

$$T_{\rm rev} = \frac{\hbar}{E_1}\left[2\pi \frac{N_n}{n^2} + \frac{\varphi}{n^2}\right]. \tag{9}$$

We will have found a revival if we can make the right-hand side independent of n. Clearly, the right-most member, φ/n^2, will be independent of n only when either 1) $\varphi = 0$ or else 2) only special values of n enter the superposition. The latter case holds, for example, when initial state is an energy eigenstate, in which case all times are revival times! (Additional examples are given in appendix B.) Turning to the case $\varphi = 0$, the ratio N_n/n^2 can be made independent of n by selecting

$$N_n = (\text{integer})n^2, \tag{10}$$

and the smallest such integer is of course one. We have proven [18]:

Theorem 1: Exact quantal revivals. Any wavefunction in an infinite square well will exactly come back to itself after a time

$$T_{\rm rev} = \frac{2\pi\hbar}{E_1} = \frac{4ML^2}{\pi\hbar}. \tag{11}$$

It is also easy to show that:

Theorem 2: Reflection half-way to a revival. After the passage of half a revival time, any wavefunction is reflected about the origin (with a physically irrelevant change of sign):

$$\psi(x;T_{\rm rev}/2) = -\psi(-x;0). \tag{12}$$

The paradox. In classical mechanics, any state comes back to itself after time $T_{\rm cl} = L\sqrt{2M/E}$, which depends on the state through the energy E. In quantum mechanics, any state comes back to itself after time $T_{\rm rev} = 4ML^2/\pi\hbar$, which is independent of state. (In both cases, the state is reflected about the origin after half a period.) How can the classical result possibly be the limit of the quantal result? In particular, since a "nearly classical" wavepacket state is just one particular type of quantal state, shouldn't it have the same periodicity?

III. Periodicity of expectation values

The quantal revival theorem gives a time after which the full wavefunction repeats. But experiments never measure the full wavefunction. If an experimental probe couples to, say, the expected position $\langle x(t)\rangle$, then what matters is the period for that $\langle x(t)\rangle$ to repeat, not the period for the full wavefunction to recur. (Other experiments probe other quantities, such as the expected momentum or the expected force. Many such quantities are related through Ehrenfest's theorem [19].) We will find that $\langle x(t)\rangle$ can have an approximate period that is a odd integral fraction of the quantal revival time.

The quantal state evolves in time as

$$|\psi(t)\rangle = \sum_{n=1}^{\infty} c_n e^{-iE_n t/\hbar}|n\rangle \tag{13}$$

so

$$\begin{aligned}
\langle x(t)\rangle &= \langle\psi(t)|x|\psi(t)\rangle \\
&= \left[\sum_{m=1}^{\infty} c_m^* e^{iE_m t/\hbar}\langle m|\right] x \left[\sum_{n=1}^{\infty} c_n e^{-iE_n t/\hbar}|n\rangle\right] \\
&= \sum_{m,n} c_m^* c_n e^{i(E_m-E_n)t/\hbar}\langle m|x|n\rangle \\
&= \sum_{m\,\text{even}}\ \sum_{n\,\text{odd}} 2\,\Re e\{c_m^* c_n e^{i(E_m-E_n)t/\hbar}\}\,\langle m|x|n\rangle,
\end{aligned} \tag{14}$$

where in the last step we used the fact (see equation 4) that $\langle m|x|n\rangle$ vanishes when m and n have the same parity.

In other words, the position $\langle x(t)\rangle$ is a Fourier sum of terms with periods of

$$\frac{2\pi\hbar}{E_m - E_n} = \frac{2\pi\hbar}{E_1(m^2-n^2)} = \frac{T_{\text{rev}}}{m^2-n^2}. \tag{15}$$

To make this result concrete, we write down some of these periods:

	$m=2$	$m=4$	$m=6$	$\cdots$
$n=1$	$T_{\text{rev}}/3$	$T_{\text{rev}}/15$	$T_{\text{rev}}/35$	
$n=3$	$T_{\text{rev}}/5$	$T_{\text{rev}}/7$	$T_{\text{rev}}/27$	$\cdots$
$n=5$	$T_{\text{rev}}/21$	$T_{\text{rev}}/9$	$T_{\text{rev}}/11$	
$\vdots$		$\vdots$		$\ddots$

Thus each term in the Fourier sum has a period *smaller* than T_{rev}. (In fact, each is an odd integral fraction of T_{rev}, as $m^2 - n^2 = (m+n)(m-n)$ is a product of odd integers and hence odd. The existence of a term with period an even integral fraction of T_{rev} would result in a violation of theorem 2: "Reflection half-way to a revival".)

For example, if the initial state is

$$|\psi\rangle = c_1|1\rangle + c_2|2\rangle + c_3|3\rangle, \tag{16}$$

where, for simplicity, we take c_1, c_2, and c_3 to be real, then the expected position is

$$\begin{aligned}\langle x(t)\rangle &= 2c_2c_1\cos((E_2-E_1)t/\hbar)\langle 2|x|1\rangle + 2c_2c_3\cos((E_2-E_3)t/\hbar)\langle 2|x|3\rangle \\ &= \frac{16L}{\pi^2}\left[\frac{2}{9}c_2c_1\cos\left(\frac{2\pi t}{T_{\rm rev}/3}\right) - \frac{6}{25}c_2c_3\cos\left(\frac{2\pi t}{T_{\rm rev}/5}\right)\right].\end{aligned} \tag{17}$$

Because 3 and 5 are relatively prime, $\langle x(t)\rangle$ is *exactly periodic* with period $T_{\rm rev}$. However, if $c_1 \gg c_3$, then $\langle x(t)\rangle$ is *almost periodic* with period $T_{\rm rev}/3$, whereas if $c_3 \gg c_1$, then it is almost periodic with period $T_{\rm rev}/5$. This example shows how $\langle x(t)\rangle$ can, depending upon the initial state, approximately cycle several times before the full wavefunction revives exactly [20].

Let us apply this understanding to a "quasiclassical" initial state, namely any superposition [21] of energy states n ranging from $n_C - r$ to $n_C + r$, with $r \ll n_C$. If i and j are integers ranging from $-r$ to $+r$, with i even and j odd, then the terms in the Fourier sum have periods of

$$\frac{T_{\rm rev}}{(n_C+i)^2-(n_C+j)^2} = \frac{T_{\rm rev}}{2n_C(i-j)+i^2-j^2} \approx \frac{T_{\rm rev}}{2n_C(i-j)}. \tag{18}$$

That is, the periods are approximately

$$\frac{T_{\rm rev}}{2n_C}, \frac{T_{\rm rev}}{2n_C(3)}, \frac{T_{\rm rev}}{2n_C(5)}, \ldots, \frac{T_{\rm rev}}{2n_C(2r-1)}. \tag{19}$$

But, of course, any function with period $T/3$ also has period T, so the Fourier sum representing the expected position $\langle x(t)\rangle$ has an overall "position period" of

$$T_{\rm pos} \equiv \frac{T_{\rm rev}}{2n_C}. \tag{20}$$

This position period is, of course, smaller than $T_{\rm rev}$ and unlike $T_{\rm rev}$ it depends on the initial state. If we define the "central energy" corresponding to the "central eigenstate", namely

$$E_C = E_1 n_C^2, \tag{21}$$

then the position period $T_{\rm pos}$ becomes

$$T_{\rm pos} = \frac{T_{\rm rev}}{2\sqrt{E_C/E_1}} = \frac{2\pi\hbar}{E_1 2\sqrt{E_C/E_1}} = \frac{\pi\hbar}{\sqrt{E_1E_C}}. \tag{22}$$

Using the ground-state energy result $E_1 = \pi^2\hbar^2/2ML^2$ (equation 1), the position period becomes

$$T_{\rm pos} = L\sqrt{\frac{2M}{E_C}}. \tag{23}$$

All factors of $\hbar$ have vanished, and this corresponds perfectly to

$$T_{\rm cl} = L\sqrt{\frac{2M}{E}}, \tag{24}$$

the classical result.

Paradox lost.

The above correspondence is the central result of this paper. Notice, first, that its derivation depends in an essential way on the energy eigenvalues, but does not actually require a knowledge of the energy eigenfunctions or of the matrix elements $\langle m|x|n\rangle$, although knowing these items rendered the derivation more concrete. Notice also that our result reinforces the old observation [11] that the classical period corresponds not to

$$\frac{2\pi\hbar}{\text{energy eigenvalue}}$$

but to

$$\frac{2\pi\hbar}{\text{energy eigenvalue spacing}}.$$

The importance of this observation has been underscored in recent years by students of quantum revivals [7, 8, 9, 10], who write the correspondence as

$$\frac{2\pi\hbar}{\left|\frac{\partial E_n}{\partial n}\right|_{n=n_C}}.$$

Another point is that the classical period $T_{\rm cl}$ is *less than* (indeed, considerably less than) the quantal period $T_{\rm rev}$. One tends to think of characteristic quantal times as short relative to characteristic classical times, but the correct idea is that characteristic times for small things are short relative to characteristic times for large things. When applied to a situation of a particular size, the characteristic quantal time can be longer than the characteristic classical time.

Finally, this result emphasizes the richness of structure available in in quantum mechanics [22]. A classical state is described completely by just two numbers: x and p. But a complete description of a quantal state requires the specification of $\langle x\rangle$, $\langle p\rangle$, Δx, Δp, $\langle xp\rangle$, and more. After a time $T_{\rm cl}$, the classical state comes back exactly on itself, while for the quantal state (if it is quasiclassical in the sense of equation 18) only $\langle x\rangle$ and $\langle p\rangle$ come back (with high accuracy) to their initial values: the other quantities can take on different values, and don't need to come back to their initial values until the longer time $T_{\rm rev}$ has passed. Thus many different quantal states can correspond to the same classical state.

The infinite square well is pathological in several ways. One is that the separation between energy eigenvalues increases as the energy grows into the supposedly-classical regime. Thus one might suspect some pathology concerning time development in the classical limit. The revival paradox lends additional weight to this feeling. This article has shown that, despite these legitimate suspicions, the classical limit of quantal time development in the infinite square well is safe and sound.

Appendix A. Expected position for a particular initial condition

This appendix considers the time evolution of $\langle x(t)\rangle$ for a quasiclassical initial Gaussian wavepacket, and goes on to uncover, not just the period, but the full triangle-wave character of motion in the classical limit.

The classical motion of a particle starting at the origin is a triangle wave of amplitude $L/2$ and period $T_{\rm cl}$, which has the Fourier series representation

$$x_{\rm cl}(t) = \frac{4L}{\pi^2} \sum_{n=1,3,5,\cdots} \frac{(-1)^{(n-1)/2}}{n^2} \sin\left(2\pi n \frac{t}{T_{\rm cl}}\right). \tag{25}$$

Suppose the initial Gaussian is

$$\psi(x) = \frac{1}{\sqrt[4]{2\pi}\sqrt{\Delta}} e^{-x^2/4\Delta^2} e^{ikx}, \tag{26}$$

which has $\langle x \rangle = 0$ and $\Delta x = \Delta$. For concreteness, we take k positive (i.e. the wavepacket initially moves to the right). This packet will be quasiclassical if it is thin compared to the width of the well,

$$\Delta \ll L, \tag{27}$$

and if the initial kinetic energy is high enough. The precise requirement, as we will see in equation (31), is that

$$k \gg L/\Delta^2. \tag{28}$$

Note that this wavefunction extends beyond the width of the square well; if the thinness condition (27) holds, this inconsistency is not fatal.

The expansion coefficients for this initial wavefunction are

$$\begin{aligned} c_n &= \langle n|\psi\rangle \\ &= \frac{1}{\sqrt[4]{2\pi}\sqrt{\Delta}}\sqrt{\frac{2}{L}}\left[\int_{-L/2}^{+L/2} \operatorname{trig}\left(\frac{\pi}{L}nx\right) e^{-x^2/4\Delta^2} e^{ikx}\, dx\right] \\ &= \frac{1}{\sqrt[4]{2\pi}\sqrt{\Delta}}\sqrt{\frac{2}{L}}\left[\int_{-\infty}^{+\infty} \operatorname{trig}\left(\frac{\pi}{L}nx\right) e^{-x^2/4\Delta^2} e^{ikx}\, dx + L\mathcal{O}(e^{-L^2/8\Delta^2})\right] \end{aligned} \tag{29}$$

where the function "trig(θ)" means $\cos(\theta)$ when n is odd and $\sin(\theta)$ when n is even. By virtue of the thinness condition (27) we ignore the terms of order $e^{-L^2/8\Delta^2}$ in what follows. The integral is straightforward and results in

$$c_n = \sqrt[4]{2\pi}\sqrt{\frac{\Delta}{L}}\frac{1}{\{i\}}\left[e^{-\Delta^2(k+\pi n/L)^2} \pm e^{-\Delta^2(k-\pi n/L)^2}\right], \tag{30}$$

where $\{i\}$ means 1 for n odd, i for n even, and where $\pm$ means $+$ for n odd, $-$ for n even.

The left exponential term will always be smaller than the right one: in fact, their ratio

$$e^{-4\pi\Delta^2 kn/L} \tag{31}$$

is tiny indeed whenever the high-energy condition (28) applies. Thus we drop the left exponential term and conclude that the expansion coefficients for our quasiclassical initial wavefunction are, to high accuracy,

$$c_n = \sqrt[4]{2\pi}\sqrt{\frac{\Delta}{L}}\,\{i\}\, e^{-\Delta^2(k-\pi n/L)^2}. \tag{32}$$

This expression has a maximum when $k - \pi n/L = 0$, whence the eigenstate most prominent in the superposition has the "central" value

$$n_C \equiv kL/\pi. \tag{33}$$

In terms of this definition the high-energy condition (28) reads

$$n_C \gg (L/\Delta)^2 \tag{34}$$

and the expansion coefficients are

$$c_n = \sqrt[4]{2\pi}\sqrt{\frac{\Delta}{L}}\,\{i\}\,e^{-(\pi\Delta/L)^2(n_C-n)^2}. \tag{35}$$

Because of the thinness condition (27) these coefficients spread over a broad range of n, but because of the high-energy condition (34) the peak of this broad distribution is so far from the origin that the ground-state coefficient c_1 is negligible.

For future use, note that the normalization condition is

$$1 = \sum_{n=1}^{\infty} |c_n|^2 = \sqrt{2\pi}\frac{\Delta}{L}\sum_{n=1}^{\infty} e^{-2(\pi\Delta/L)^2(n_C-n)^2}. \tag{36}$$

But the summand $e^{-2(\pi\Delta/L)^2(n_C-n)^2}$ is utterly negligible for $n = 0, -1, -2, \ldots$, so the lower summation limit can be extended down to $-\infty$, and once this is done the summation index can be shifted to $n' = n - n_C$ to conclude that

$$1 = \sqrt{2\pi}\frac{\Delta}{L}\sum_{n'=-\infty}^{+\infty} e^{-2(\pi\Delta/L)^2 n'^2}, \tag{37}$$

or, in general, that

$$\sum_{n'=-\infty}^{+\infty} e^{-2\alpha^2 n'^2} = \frac{1}{\alpha}\sqrt{\frac{\pi}{2}} \tag{38}$$

for sufficiently small values of α.

To employ equation (14) for $\langle x(t)\rangle$, we need to find

$$\Re e\{c_m^* c_n e^{i(E_m-E_n)t/\hbar}\} \tag{39}$$

for m even, n odd. Using equation (35), the result is

$$\sqrt{2\pi}\frac{\Delta}{L}e^{-(\pi\Delta/L)^2[(n_C-m)^2+(n_C-n)^2]}\sin(E_1(m^2-n^2)t/\hbar). \tag{40}$$

Then, using equations (40) and (4) in the time evolution equation (14), we have

$$\begin{aligned}\langle x(t)\rangle &= \sum_{m\,\text{even}}\sum_{n\,\text{odd}} 2\,\Re e\{c_m^* c_n e^{i(E_m-E_n)t/\hbar}\}\,\langle m|x|n\rangle \\ &= \sqrt{2\pi}\frac{\Delta}{L}\frac{16L}{\pi^2}\sum_{m\,\text{even}}\sum_{n\,\text{odd}} e^{-(\pi\Delta/L)^2[(n_C-m)^2+(n_C-n)^2]}\sin(E_1(m^2-n^2)t/\hbar)\ \times \\ &\qquad (-1)^{(m-n-1)/2}\frac{mn}{(m^2-n^2)^2}.\end{aligned} \tag{41}$$

(We have substituted $(-1)^{(m-n-1)/2}$ for $(-1)^{(m+n+1)/2}$, which is legitimate because n is odd.)

Because of the exponential factor, the summand is negligible except when m and n are close to n_C. This inspires the definitions

$$m' = m - n_C \quad \text{and} \quad n' = n - n_C. \tag{42}$$

In the region where the summand is not negligible, we have

$$m^2 - n^2 = (n_C + m')^2 - (n_C + n')^2 = 2n_C(m' - n') + m'^2 - n'^2 \approx 2n_C(m' - n') \tag{43}$$

and

$$\frac{mn}{(m^2-n^2)^2} \approx \frac{n_C^2}{4n_C^2(m'-n')^2} = \frac{1}{4(m'-n')^2}. \tag{44}$$

Furthermore, we may extend the summation range down to $-\infty$, and shift the summation indices from m and n to m' and n'. Assuming that n_C is even, this results in

$$\langle x(t)\rangle = \sqrt{2\pi}\frac{4\Delta}{\pi^2} \sum_{m'=0,\pm2,\cdots} \sum_{n'=\pm1,\pm3,\cdots} e^{-(\pi\Delta/L)^2[m'^2+n'^2]} \sin\left(\frac{E_1 2n_C(m'-n')t}{\hbar}\right) \frac{(-1)^{(m'-n'-1)/2}}{(m'-n')^2}. \tag{45}$$

(If n_C were odd, then m' would be odd and n' even.)

Using equation (20), we recognize the argument of the sine function as $(2\pi(m'-n')t/T_{\text{pos}})$. Also, in a maneuver of surprising subtlety, we change the summation indices to

$$j = m' + n' \quad \text{and} \quad \ell = m' - n' \tag{46}$$

where ℓ ranges over all odd integers and j takes on the values $-\ell$, $-\ell \pm 4$, $-\ell \pm 8$, ... The result is

$$\langle x(t)\rangle = \sqrt{2\pi}\frac{4\Delta}{\pi^2} \sum_{\ell=\pm1,\pm3,\cdots} \sum_{j} e^{-(\pi\Delta/L)^2[j^2+\ell^2]/2} \sin\left(\frac{2\pi\ell t}{T_{\text{pos}}}\right) \frac{(-1)^{(\ell-1)/2}}{\ell^2}, \tag{47}$$

or

$$\langle x(t)\rangle = \sqrt{2\pi}\frac{4\Delta}{\pi^2} \sum_{\ell=\pm1,\pm3,\cdots} \left[\sum_{j} e^{-(\pi\Delta/L)^2 j^2/2}\right] e^{-(\pi\Delta/L)^2\ell^2/2} \sin\left(\frac{2\pi\ell t}{T_{\text{pos}}}\right) \frac{(-1)^{(\ell-1)/2}}{\ell^2}. \tag{48}$$

The normalization-derived result (38) tells us that the sum in square brackets is

$$\sum_{j} e^{-(\pi\Delta/L)^2 j^2/2} = \frac{1}{4}\sum_{j=-\infty}^{+\infty} e^{-(\pi\Delta/L)^2 j^2/2} = \frac{1}{4}\frac{2L}{\pi\Delta}\sqrt{\frac{\pi}{2}}, \tag{49}$$

so

$$\langle x(t)\rangle = \frac{2L}{\pi^2} \sum_{\ell=\pm1,\pm3,\cdots} e^{-(\pi\Delta/L)^2\ell^2/2} \sin\left(\frac{2\pi\ell t}{T_{\text{pos}}}\right) \frac{(-1)^{(\ell-1)/2}}{\ell^2}. \tag{50}$$

The same result obtains whether n_C is even or odd.

Finally, and for the last time, we use the thinness condition (27) to let all the exponential arguments tend to zero and all the exponentials tend to one, whence

$$\langle x(t)\rangle = \frac{2L}{\pi^2} \sum_{\ell=\pm1,\pm3,\cdots} \sin\left(\frac{2\pi\ell t}{T_{\text{pos}}}\right) \frac{(-1)^{(\ell-1)/2}}{\ell^2}. \tag{51}$$

The summand is unchanged if ℓ is replaced by $-\ell$ (for ℓ an odd integer), so

$$\langle x(t)\rangle = \frac{4L}{\pi^2}\sum_{\ell=1,3,5,\cdots} \sin\left(\frac{2\pi\ell t}{T_{\text{pos}}}\right)\frac{(-1)^{(\ell-1)/2}}{\ell^2}. \tag{52}$$

All references to Δ have vanished (as long as $\Delta \ll L$) and we recover the classical result (25).

Appendix B. State-dependent revival times

Theorem 1 (equation 11) gives a universal revival time T_{rev} after which *any* wavefunction comes back to itself. But certain special wavefunctions revive at an integer fraction of T_{rev}, as detailed in this appendix.

Theorems 3, 4, and 5 here are easy to prove, so their proofs are not presented. Theorem 6 is more difficult and more general (indeed, theorems 1, 3, 4, and 5 are all special cases of theorem 6). It is also worth noting that theorems 1, 3, and 6 rely only upon the fact that the energy eigenvalues are proportional to n^2, and proving them does not require any knowledge of the energy eigenfunctions. Theorems 2, 4, and 5, in contrast, depend upon the parity of the energy eigenfunctions, although not on their precise form.

Theorem 3: Mock Rabi oscillations. If the wavefunction is a superposition of only two energy eigenfunctions, namely n and m, then it revives after the shorter time

$$\tau_{n,m} = \frac{T_{\text{rev}}}{m^2-n^2} = \frac{2\pi\hbar}{E_m - E_n}, \tag{53}$$

at which time

$$\psi(x;\tau_{n,m}) = e^{-2\pi i n^2/(m^2-n^2)}\psi(x;0) = e^{-2\pi i m^2/(m^2-n^2)}\psi(x;0). \tag{54}$$

(Note the physically irrelevant change of phase.)

Theorem 4. If a wavefunction has odd parity, it revives after the passage of time $T_{\text{rev}}/4$:

$$\psi_o(x;T_{\text{rev}}/4) = \psi_o(x;0). \tag{55}$$

Theorem 5. If a wavefunction has even parity, it revives after the passage of time $T_{\text{rev}}/8$:

$$\psi_e(x;T_{\text{rev}}/8) = e^{-i\pi/4}\psi_e(x;0). \tag{56}$$

Theorem 6. If a wavefunction is a superposition of a finite number energy eigenstates, namely n_a, n_b, n_c, n_d, $\ldots$, n_z, then its first revival comes at time

$$\tau = \frac{T_{\text{rev}}}{\text{GCD}[n_b^2-n_a^2, n_c^2-n_a^2, n_d^2-n_a^2, \cdots, n_z^2-n_a^2]}, \tag{57}$$

were "GCD" signifies the greatest common divisor.

Proof of theorem 6: The wavefunction is a superposition of energy eigenstates with energies (listed from smallest to largest)

$$E_a,\, E_b,\, E_c,\, E_d, \ldots,\, E_z. \tag{58}$$

Then the quantal state will recur exactly at a time τ when the phase factors

$$e^{-iE_a\tau/\hbar},\, e^{-iE_b\tau/\hbar},\, e^{-iE_c\tau/\hbar},\, e^{-iE_d\tau/\hbar}, \ldots,\, e^{-iE_z\tau/\hbar} \tag{59}$$

are all equal. We search for the shortest such time.

Analogy: Think of each of the phase factors (59) as a phasor clock with a single hand. Each hand sweeps out a circle periodically, but the clocks are poorly calibrated so each hand circles with a different period. Our question is: If the clock hands all start out pointing vertically, how long will it take before they are all parallel again? Quantal revival, in other words, is just an elaboration upon the old joke that a stopped clock gives the correct time twice a day.

First we find the shortest time τ_ℓ for the two phase factors

$$e^{-iE_a\tau_\ell/\hbar} \quad \text{and} \quad e^{-iE_\ell\tau_\ell/\hbar} \tag{60}$$

to be equal. This happens when both

$$\frac{E_a}{\hbar}\tau_\ell = 2\pi N_a + \varphi \tag{61}$$

and

$$\frac{E_\ell}{\hbar}\tau_\ell = 2\pi N_\ell + \varphi. \tag{62}$$

Taking the difference gives

$$\frac{E_\ell - E_a}{\hbar}\tau_\ell = 2\pi(N_\ell - N_a), \tag{63}$$

and the smallest such τ_ℓ comes when $N_\ell - N_a = 1$, so

$$\tau_\ell = \frac{2\pi\hbar}{E_\ell - E_a}. \tag{64}$$

This gives the smallest time for clock hands a and ℓ to be parallel. For *all* the clock hands to be parallel, we need a and b to be parallel, and the same for a and c, a and d, and so forth up to a and z. That is, we need positive integers N_b through N_z such that

$$N_b\tau_b = N_c\tau_c = N_d\tau_d = \cdots = N_z\tau_z. \tag{65}$$

If these integers have no common factors, then the resulting time is the smallest revival time τ.

Up to this point we have not specified the energies so our results apply equally to any quantal system with discrete spectrum, not just to the infinite square well. (Indeed, we are only a few lines from a proof of the general "quantum revival theorem" [6].) We now specialize to the infinite square well case $E_n = E_1 n^2$.

In this situation the revival time for the pair consisting of a and ℓ is

$$\tau_\ell = \frac{2\pi\hbar}{E_\ell - E_a} = \frac{2\pi\hbar}{E_1(n_\ell^2 - n_a^2)} = \frac{T_{\text{rev}}}{n_\ell^2 - n_a^2}. \tag{66}$$

Equation (65) becomes

$$\tau = N_b \tau_b = N_c \tau_c = N_d \tau_d = \cdots = N_z \tau_z \tag{67}$$

or

$$\frac{\tau}{T_{\mathrm{rev}}} = \frac{N_b}{n_b^2 - n_a^2} = \frac{N_c}{n_c^2 - n_a^2} = \frac{N_d}{n_d^2 - n_a^2} = \cdots = \frac{N_z}{n_z^2 - n_a^2}. \tag{68}$$

If we pick $N_b = n_b^2 - n_a^2$ and so forth, then we come up with $\tau = T_{\mathrm{rev}}$. But is there a shorter revival time? There is if the N_ℓ integers so chosen possess a common factor. When that happens, we can divide out this common factor (the greatest common divisor) to find the minimum revival time

$$\tau = \frac{T_{\mathrm{rev}}}{\mathrm{GCD}[n_b^2 - n_a^2, n_c^2 - n_a^2, n_d^2 - n_a^2, \cdots, n_z^2 - n_a^2]}, \tag{69}$$

and the theorem is proved. ∎

In brief, some particular quantum states have revival times less than the universal quantum revival time T_{rev}, which is the time at which time *all* quantal states revive. (Clearly the state-specific revival times must be integral fractions of the universal revival time.)

Lest a misconception arise, however, I emphasize that this phenomena is *not* responsible for the paradox at the center of this paper, namely the classical periodicity being less than the universal quantal revival time. In fact, the quasiclassical tight wavepackets considered just above equation (18) consist of superpositions of $2r+1$ adjacent energy eigenstates, and it is easy to prove that for three or more adjacent energy eigenstates, the GCD of equation (69) is equal to one.

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References

[1] P. Ehrenfest, "Bemerkung über die angenäherte Gültigkeit der klassischen Mechanik innerhalb der Quantenmechanik", Z. Physik **45**, 455–457 (1927). Reprinted in Martin J. Klein, editor, *Paul Ehrenfest: Collected Scientific Papers* (North-Holland, Amsterdam, 1959) pages 556–558.

[2] Michael Berry, "Some quantum-to-classical asymptotics", in M.-J. Giannoni, A. Voros, and J. Zinn-Justin, editors, *Chaos and Quantum Physics* (Les Houches 1989) (North-Holland, Amsterdam, 1991) pages 250–303.

[3] W.H. Zurek, "Decoherence and the transition from quantum to classical", Physics Today **44** (10), 36–44 (October 1991). Serge Haroche, "Entanglement, decoherence and the quantum/classical boundary", Physics Today **51** (7), 36–42 (July 1998).

[4] Martin C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, New York, 1990).

[5] Eric J. Heller and Steven Tomsovic, "Postmodern quantum mechanics", Physics Today **46** (7), 38–46 (July 1993).

[6] R. Bocchieri and A. Loinger, "Quantum recurrence theorem", Phys. Rev. **107**, 337–338 (1957).

[7] James Bayfield, *Quantum Evolution: An Introduction to Time-Dependent Quantum Mechanics* (Wiley, New York, 1999).

[8] Tamar Seideman, "Revival structure of aligned rotational wave packets", Phys. Rev. Lett. **83**, 4971–4974 (1999).

[9] Robert Bluhm, V. Alan Kostelecký, and James A. Porter, "The evolution and revival structure of localized quantum wave packets", Am. J. Phys. **64**, 944–953 (1996).

[10] David L. Aronstein and C.R. Stroud, Jr., "Fractional wave-function revivals in the infinite square well," Phys. Rev. A **55**, 4526–4537 (1997).

[11] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory* (Pergamon Press, London, 1958) particularly §48 and equation (48.5) on page 165.

[12] David K. Ferry and Stephen M. Goodnick, *Transport in Nanostructures* (Cambridge University Press, Cambridge, UK, 1997).

[13] Elmar Schreiber, *Femtosecond Real-time Spectroscopy of Small Molecules and Clusters* (Springer, Berlin, 1998).

[14] J. Feldmann, T. Meier, G. von Plessen, M. Koch, E.O. Göbel, P. Thomas, G. Bacher, C. Hartmann, H. Schweizer, W. Schäfer, and H. Nickel, "Coherent dynamics of excitonic wave packets", Phys. Rev. Lett. **70**, 3027–3030 (1993).

[15] A. Bonvalet, J. Nagle, V. Berger, A. Migus, J.-L. Martin, and M. Joffre, "Femtosecond infrared emission resulting from coherent charge oscillations in quantum wells", Phys. Rev. Lett. **76**, 4392–4395 (1996).

[16] R.L. Liboff, "The correspondence principle revisited", Physics Today **37** (2), 50–55 (February 1984).

[17] I represent an arbitrary wavefunction by $\psi(x)$ and an energy eigenfunction by $\eta(x)$, following the admirable convention established by Daniel T. Gillespie in *A Quantum Mechanics Primer* (International Textbook Company, Scranton, Pennsylvania, 1970).

[18] This result has been known for some time and rediscovered frequently. The earliest printed mention that I can find is Carlo U. Segre and J.D. Sullivan, "Bound-state wave packets", Am. J. Phys. **44**, 729–732 (1976). See also references [9] and [10], and Siegmund Brandt and Hans Dieter Dahmen, *The Picture Book of Quantum Mechanics*, second edition (Springer-Verlag, Berlin, 1995) pages 107–108.

[19] The applicability of Ehrenfest's theorem to the pathological infinite square well is treated in the delightful paper D.S. Rokhsar, "Ehrenfest's theorem and the particle-in-the-box", Am. J. Phys. **64**, 1416–1418 (1996).

[20] If the initial state is, for example, a superposition of the energy eigenstates $|1\rangle$, $|2\rangle$, and $|5\rangle$, then $\langle x(t)\rangle$ is a sum of one term with period $T_{\rm rev}/3$ and another with period $T_{\rm rev}/21$, whence $\langle x(t)\rangle$ is exactly periodic with period $T_{\rm rev}/3$. Remarkably, it turns out that in all such cases the full wavefunction recurs with precisely the same period: see theorem 6 in appendix B.

[21] Notice that such a quasiclassical state does not need to be a conventional wavepacket, although all conventional wavepackets are quasiclassical states. The term "wavepacket" usually refers to a wavefunction that is well-localized in space, with a probability density that decreases from the center. The term is also used for a wavefunction that is well-defined in energy, with an energy probability that decreases from the central energy. A quasiclassical state as defined here need not be well-localized in space, and, while it is a superposition of energy states in a particular range, within that range the energy probability may vary in any imaginably way.

[22] D.F. Styer, *The Strange World of Quantum Mechanics* (Cambridge University Press, Cambridge, UK, 2000) appendix B.2.